Lecture 5:

Theorem: Suppose S is a finite spanning set for a vector space V.

Then: $\exists\, \beta \subset S$ which is a basis for V.

(A finite spanning set can be reduced to a basis)

Proof: If S is lin. independent, then we take $\beta = S$.

Otherwise, $\exists\, \vec{v}_1 \in S$ such that $\mathrm{Span}(S \setminus \{\vec{v}_1\}) = \mathrm{Span}(S)$ (by lemma)

If $S \setminus \{\vec{v}_1\}$ is linearly independent, then take $\beta = S \setminus \{\vec{v}_1\}$.

Otherwise, $\exists\, \vec{v}_2 \in S \setminus \{\vec{v}_1\}$ such that $\mathrm{Span}(S \setminus \{\vec{v}_1, \vec{v}_2\}) = \mathrm{Span}(S \setminus \{\vec{v}_1\})$

Repeat this process.

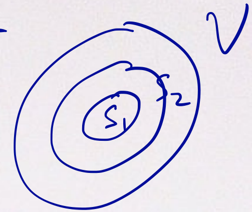
$\because$ S is finite. The process must stop at a linearly independent subset $S_k = S \setminus \{\vec{v}_1, \dots, \vec{v}_k\} \subset S$ and $\mathrm{Span}(S_k) = \underset{V}{\underset{\shortparallel}{\mathrm{Span}(S)}}$

Take $\beta = S_k$.

<u>Lemma</u>: Let V be a vector space, and let $S_1 \subset S_2 \subset V$. Then:

(a) S_1 is linearly dependent $\Rightarrow$ S_2 is linearly dependent

S_1 is linearly independent $\Leftarrow$ S_2 is linearly independent

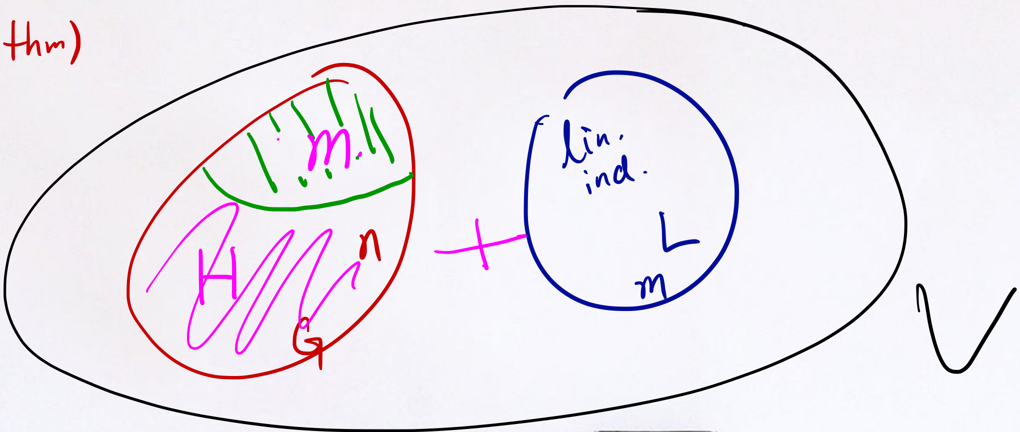


(b) $\mathrm{Span}(S_1) \subset \mathrm{Span}(S_2)$

<u>Proof</u>: Exercise.

Theorem: Let V be a vector space.
Let $G \subset V$ be a spanning set for V consisting of n vectors.
and $L \subset V$ be a linearly independent subset consisting of m vectors.
Then, $m \leq n$ and $\exists\, H \subset G$ consisting of exactly $n-m$ vectors
such that $L \cup H$ spans V.

(Replacement thm)



Proof: We prove by induction on $m \geq 0$

For $m=0$, $L = \phi$. Then: $m \leq n$. Also, take $H = G$.

Suppose the statement is true for some $m \geq 0$. We need to show that the statement is also true for $m+1$.

So, let $L = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m+1}\}$ be a linearly independent subset of V.

Then: $L' = \{\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m}_{m \text{ elements.}}\} \subset L$ is linearly independent.

By induction hypothesis, we have $m \leq n$ and

$\exists H' = \{\vec{u}_1, \ldots, \vec{u}_{n-m}\} \subset G$ such that

$$L' \cup H' = \{\vec{v}_1, \ldots, \vec{v}_m, \vec{u}_1, \ldots, \vec{u}_{n-m}\} \text{ span } V.$$

In particular, $\exists\ a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{n-m} \in F$ such that

$$\vec{v}_{m+1} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m + b_1\vec{u}_1 + \ldots + b_{n-m}\vec{u}_{n-m}.$$

But $L = \{\vec{v}_1, \ldots, \vec{v}_{m+1}\}$ is linearly independent. So, $n-m \geq 1$

or $\boxed{m+1 \leq n}$

and one of b_k's, say b_1, is non-zero

This implies, $\vec{u}_1 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m+1}, \vec{u}_2, \ldots, \vec{u}_{n-m}\}$

$\therefore$ Take $H \overset{\text{def}}{:=} \{\vec{u}_2, \ldots, \vec{u}_{n-m}\}$.

Then $\underbrace{L}_{m+1} \cup \underbrace{H}_{n-(m+1)}$ span V

This completes the induction argument.

Dimension

Cor 1: Let V be a vector space having a finite basis. Then, every basis of V contains the same number of vectors.

Pf: Let β and γ be two bases of V.

Since β spans V and γ is lin. independent,

then $|\gamma| \leq |\beta|$ (by replacement Thm)

Similarly, $|\beta| \leq |\gamma|$

$\Rightarrow |\gamma| = |\beta|$.

Definition: A vector space V is called **finite-dimensional** if it has a finite basis. The **dimension** of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-dimensional is called **infinite-dimensional**.