Lecture 21:

Recall:

- $\overset{\text{linear}}{g} : V \to F$, $\exists !\, \vec{y} \in V \ni$ (such that) $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for $\forall \vec{x}$.

- $T : V \to V$ (linear), define $T^* : V \to V \ni$

$$\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle \quad \forall \vec{x}, \vec{y} \in V.$$

- $$[T^*]_\beta = ([T]_\beta)^* \leftarrow \text{conjugate transpose}$$

adjoint (T^*), orthonormal basis (β)

Corollary: Let A be an $n \times n$ matrix. Then: $L_{A^*} = (L_A)^*$

adjoint

conjugate transpose

Pf: The standard basis β for F^n is orthonormal.

Then: $[L_A]_\beta = A$.

$\therefore [(L_A)^*]_\beta = ([L_A]_\beta)^* = A^* = [L_{A^*}]_\beta \Rightarrow (L_A)^* = L_{A^*}$.

Proposition: Let V be an inner product space. Let $T, U = V \to V$.

Then: (a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\, T^* \quad \forall c \in F$

(c) $(TU)^* = U^* T^*$

(d) $(T^*)^* = T$

(e) $I^* = I$

Proof: $\forall \vec{x}, \vec{y} \in V$

(a) $< \vec{x}, (T+U)^*(\vec{y}) > = <(T+U)(\vec{x}), \vec{y}> = < T(\vec{x}), \vec{y} > + < U(\vec{x}), \vec{y} >$

$= < \vec{x}, T^*(\vec{y}) > + < \vec{x}, U^*(\vec{y}) >$

$= < \vec{x}, (T^* + U^*)(\vec{y}) >$

$\Rightarrow (T+U)^* = T^* + U^*$.

(b) $\langle \vec{x}, (cT)^*(\vec{y}) \rangle = \langle cT(\vec{x}), \vec{y} \rangle$

$$= c \langle T(\vec{x}), \vec{y} \rangle$$

$$= c \langle \vec{x}, T^*(\vec{y}) \rangle = \langle \vec{x}, \bar{c}\, T^*(\vec{y}) \rangle$$

$\therefore \ (cT)^* = \bar{c}\, T^*$

(c) $\langle \vec{x}, (TU)^*(\vec{y}) \rangle = \langle T(U(\vec{x})), \vec{y} \rangle$

$$= \langle U(\vec{x}), T^* \vec{y} \rangle$$

$$= \langle \vec{x}, U^* T^* \vec{y} \rangle$$

$\Rightarrow \ (TU)^* = U^* T^*.$

(d) $\langle \vec{x}, T(\vec{y}) \rangle = \langle T^*(\vec{x}), \vec{y} \rangle = \langle \vec{x}, (T^*)^*(\vec{y}) \rangle$

$\Rightarrow \quad T = T^{**}.$

(e). follows from the definition.

$$\langle \vec{x}, I(\vec{y}) \rangle = \langle I(\vec{x}), \vec{y} \rangle$$

$$\langle \vec{x}, I(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$$

<u>Remark:</u> Let A and B be $n \times n$ matrices. Then:

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$.

Lemma: Let $T: V \to V$ be a linear operator on a finite-dim inner product space V. If T has an eigenvector, then so does T^*.

Pf: Suppose $\vec{v} \in V \setminus \{\vec{0}\}$ is an eigenvector of T with eigenvalue λ.

Then: $\forall \vec{x} \in V$, we have:

$$0 = \langle \vec{0}, \vec{x} \rangle = \langle (T - \lambda I)(\vec{v}), \vec{x} \rangle = \langle \vec{v}, \underbrace{\overbrace{(T - \lambda I)^*}^{T^* - \bar{\lambda} I}(\vec{x})}_{\in R(T^* - \bar{\lambda} I)} \rangle$$

$\Rightarrow \vec{v} \in R(T^* - \bar{\lambda} I)^{\perp}$. So, $\dim(R(T^* - \bar{\lambda} I)) < \dim(V)$.

$(\dim(W) + \dim(W^{\perp}) = \dim(V))$

$\Rightarrow \dim(N(T^* - \bar{\lambda} I)) > 0$ $\therefore$ T^* has an eigenvector with eigenvalue $\bar{\lambda}$.

Thm (Schur) Let T be a lin. operator on a finite-dim inner product space. Suppose the char. poly of T splits.

Then: $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

Pf: We prove by induction on $n = \dim(V)$.

The $n=1$ case is obvious.

Assume the statement holds for lin. operators defined on $(n-1)$-dim inner product space, whose char. poly splits.

By lemma, T^* has a unit eigenvector $\vec{z}$.

Let $W \stackrel{\text{def}}{:=} \text{span}\{\vec{z}\}$ and suppose $T^*(\vec{z}) = \lambda \vec{z}$.

Claim: $W^\perp$ is T-invariant.

Pf: Let $\vec{y} \in W^\perp$ and $\vec{x} = c\vec{z} \in W$. Then:

$$\langle T(\vec{y}), \vec{x} \rangle = \langle T(\vec{y}), c\vec{z} \rangle = \langle \vec{y}, cT^*(\vec{z}) \rangle$$
$$= \langle \vec{y}, c\lambda \vec{z} \rangle$$
$$= \bar{c}\bar{\lambda} \langle \underset{W^\perp}{\vec{y}}, \underset{W}{\vec{z}} \rangle = 0$$

$\therefore\ T(\vec{y}) \in W^\perp$.

Now, $f_{T_{W^\perp}}(t) \mid f_T(t) \Rightarrow f_{T_{W^\perp}}(t)$ splits. ①

Also, $\dim(W^\perp) = n-1$ ②

$\therefore$ Induction hypothesis gives an orthonormal basis γ for $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Then: $\beta \stackrel{def}{=} \gamma \cup \{\vec{z}\}$ is orthonormal basis s.t.

$\gamma \subseteq W^\perp$, $\quad \{\vec{z}\} \subseteq W$

$$[T]_\beta = \begin{pmatrix} [T_{W^\perp}]_\gamma & \begin{matrix} * \\ * \\ * \\ * \\ \vdots \end{matrix} \\ 0 \; \cdots \; 0 & * \end{pmatrix}$$

is upper triangular