Lecture 20:

Recap:

Proposition: Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can extended to an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

(b) If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$

(c) If W is any subspace of V, then:

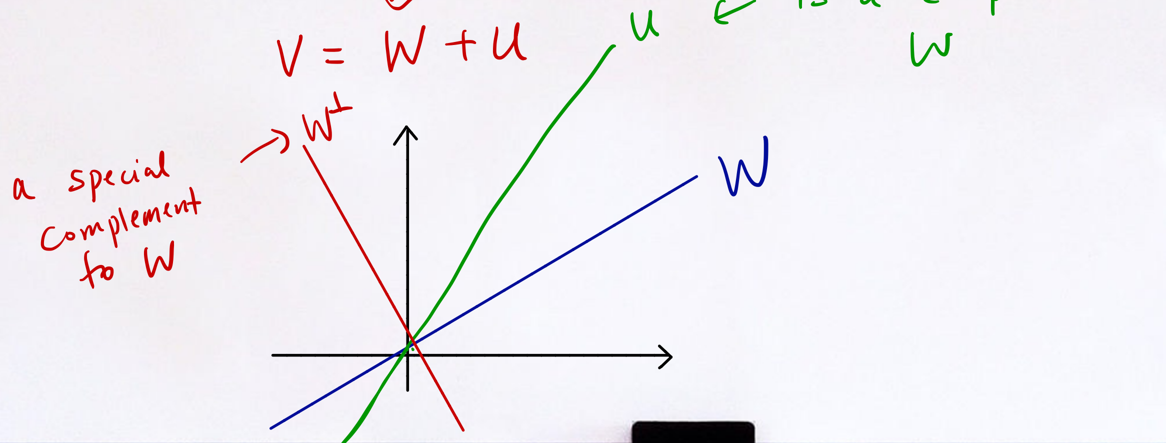
$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Remark: In fact, a "complement" to a subspace $W \subset V$ ($\dim(V) < \infty$) is another subspace $U \subset V$ s.t.

$$\begin{cases} \cdot\ W \cap U = \{\vec{0}\} \\ \cdot\ \dim(W) + \dim(U) = \dim(V) \end{cases} \Longrightarrow V = W \oplus U$$

$$\Downarrow$$

$$V = W + U$$



Adjoint of a linear operator

Prop: Let V be a finite-dim. inner product space over F.
Then for any linear transformation $g : V \to F$ (linear functional),
$\exists ! \, \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof: Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthornormal basis for V.

Set :
$$\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)} \, \vec{v}_i \, .$$

We have :
$$\langle \vec{v}_j, \vec{y} \rangle = \sum_{i=1}^{n} g(\vec{v}_i) \langle \vec{v}_j, \vec{v}_i \rangle = g(\vec{v}_j)$$

$$\Rightarrow \quad g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle \quad \text{for all } \vec{x} \in V$$

$g(\cdot)$ and $\langle \cdot , y \rangle$ agrees ~~at~~ on all basis element

If $\exists \vec{y}' \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall \vec{x}$.

then, $\langle \vec{x}, \vec{y} \rangle = g(\vec{x}) = \langle \vec{x}, \vec{y}' \rangle$ for $\forall \vec{x}$

$$\Rightarrow \quad \vec{y} = \vec{y}'.$$

Theorem: Let V be a finite-dim inner product space. Let T be a linear operator on V. Then: $\exists!$ linear operator $T^*: V \to V$ such that: $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$.

T^* is called the adjoint of T.

Proof: Given any $\vec{y} \in V$, the map $g_{\vec{y}}: V \to F$ defined by:

$$g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$$

is linear ($\because$ $\langle \cdot, \cdot \rangle$ is linear in the 1^{st} argument + T is linear)

By the previous proposition, $\exists!\ \vec{y}\,' \in V$ such that

$$\underset{\substack{\parallel \\ \vec{g}_{\vec{y}}(\vec{x})}}{\langle T(\vec{x}), \vec{y} \rangle} = \langle \vec{x}, \vec{y}\,' \rangle \text{ for all } \vec{x} \in V.$$

Now, we uniquely define: $T^*: V \to V$ by $T^*(\vec{y}) = \vec{y}\,'$.

To see that T^* is linear, let $\vec{y}_1, \vec{y}_2 \in V$ and $c \in F$.

Then $\forall \vec{x} \in V$, we have:

$$\begin{aligned}
\langle \vec{x}, T^*(c\vec{y}_1 + \vec{y}_2) \rangle &= \langle T(\vec{x}), c\vec{y}_1 + \vec{y}_2 \rangle \\
&= \bar{c} \langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle \\
&= \bar{c} \langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle \\
&= \langle \vec{x}, c\,T^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle
\end{aligned}$$

$$\Rightarrow \quad T^*(c\vec{y}_1 + \vec{y}_2) = c\,T^*(\vec{y}_1) + T^*(\vec{y}_2)$$

Remark:

$$\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle$$

Proposition: Let V be a finite-dim inner product space and let β be an orthonormal basis for V. Then $\forall\, T: V \to V$, we have:

$$[T^*]_\beta = \left([T]_\beta\right)^* \leftarrow \text{conjugate tranpose}$$

$$\left(A^* = \left(\overline{A}\right)^T \right)$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

$$B_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle = \langle \vec{v}_j, T(\vec{v}_i) \rangle = \overline{\langle T(\vec{v}_i), \vec{v}_j \rangle}$$

$$= \overline{A_{ji}} \quad //$$

Corollary: Let A be an $n \times n$ matrix. Then: $L_{A^*} = (L_A)^*$

adjoint

conjugate transpose

Pf: The standard basis β for F^n is orthonormal.

Then: $[L_A]_\beta = A$.

$$\therefore \left[(L_A)^*\right]_\beta = \left([L_A]_\beta\right)^* = A^* = [L_{A^*}]_\beta \Rightarrow (L_A)^* = L_{A^*}$$