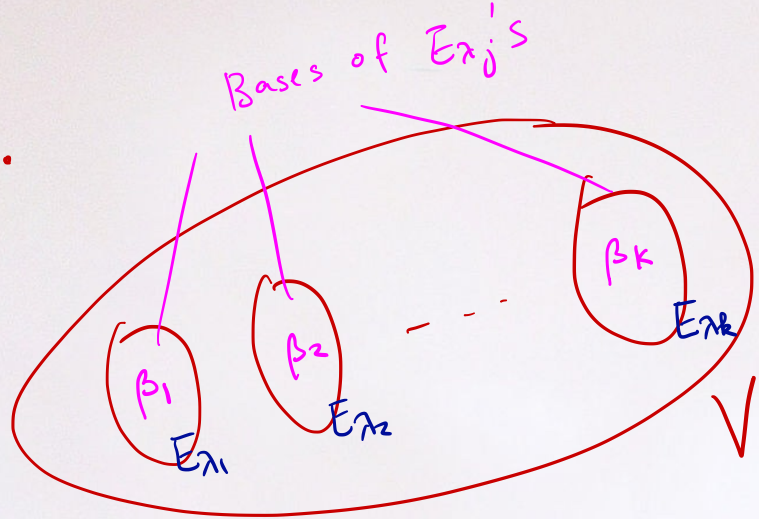
Lecture 15

Recall: $T: V \to V$, $F = \mathbb{C}$.

Char poly of T : $f_T(t) = (-1)^n (t-\lambda_1)^{n_1} (t-\lambda_2)^{n_2} \cdots (t-\lambda_k)^{n_k}$

n_j = algebraic multiplicity of $\lambda_j = \mu_T(\lambda_j)$

$$\underbrace{\dim(\text{Eigenspace of } \lambda_j) = \dim\big(N(T - \lambda_j I_V)\big) = \dim(E_{\lambda_j})}_{\text{Geometric multiplicity } = \gamma_T(\lambda_j)}$$

- T is diagonalizable iff $\mu_T(\lambda_j) = \gamma_T(\lambda_j)$ for $j = 1, 2, \dots, k$
- 

Then: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is a basis of eigenvectors for V.

Theorem: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial $f_T(t)$ splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T.

Then: (a) T is diagonalizable iff: $\mu_T(\lambda_i) = \gamma_T(\lambda_i)$ for $i = 1, 2, \ldots, k$

(b) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then: $\beta := \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors.

(so that $[T]_\beta$ is a diagonal matrix)

<u>Proof:</u> Write $n = \dim(V)$, and $m_i = \mu_T(\lambda_i)$ and $d_i = \underset{\dim(E_{\lambda_i})}{\underset{\shortparallel}{\gamma_T(\lambda_i)}}$ for all i.

Suppose T is diagonalizable and β is a basis for V consisting of eigenvectors of T.

(e.g. $\beta = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \ldots, \vec{v}_n \}$)

E_{λ_1} $\qquad$ E_{λ_j}

For each i, let $\beta_i = \beta \cap E_{\lambda_i}$ and $n_i \overset{\text{def}}{=} \# \beta_i$

Then: $n_i \leq d_i = \dim(E_{\lambda_i})$ ($\because$ β_i is lin. independent)

Also, $d_i \leq m_i$ (last lecture)

So, we have $n_i \leq d_i \leq m_i$ for all i.

$$\therefore \quad n = \sum_{i=1}^{k} n_i \le \sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} m_i = n = \dim(V)$$

$$\therefore \quad \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} n_i = 0 \iff \sum_{i=1}^{k} (d_i \overset{1,0}{-} n_i) = 0$$

$$\Rightarrow \quad d_i = n_i \quad \text{for all } i.$$

$$\therefore \quad \sum_{i=1}^{k} m_i - \sum_{i=1}^{k} d_i = 0 \iff \sum_{i=1}^{k} (m_i \overset{1,0}{-} d_i) = 0$$

$$\Rightarrow \quad d_i = m_i \quad \text{for all } i.$$

$$\therefore \quad n_i = \overset{\dim(E_{\lambda_i})}{\overset{\|}{d_i}} = m_i \quad \text{for all } i$$

(So, β_i is a basis of E_{λ_i})

Conversely, suppose $m_i = d_i \ \forall i$.

For each i, let β_i be the ordered basis of E_{λ_i}

and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$.

Then: from previous proposition, we know β is linearly independent.

But $\#\beta = \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n = \dim(V)$

$$\#\beta = |\beta_1| + |\beta_2| + \cdots + |\beta_k|$$

$$|\beta_1| = \dim(E_{\lambda_1}) = d_1, \quad |\beta_2| = \dim(E_{\lambda_2}) = d_2, \quad |\beta_k| = \dim(E_{\lambda_k}) = d_k$$

$\therefore$ β is a basis for V of eigenvectors

$\therefore$ T is diagonalizable.

__Example:__ Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by:

$$T(f(x)) = f(x) + (x+1)f'(x)$$

Then: $A := [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

where $\beta = \{1, x, x^2\}$ = standard ordered basis for $P_2(\mathbb{R})$.

$\therefore$ the char. poly:

$$\det(A - tI_3) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = (1-t)^1(2-t)^1(3-t)^1$$

$$\left.\begin{array}{l} 1 \leq \gamma_T(1) \leq \mu_T(1) = 1 \\ 1 \leq \gamma_T(2) \leq \mu_T(2) = 1 \\ 1 \leq \gamma_T(3) \leq \mu_T(3) = 1 \end{array}\right\} \quad \begin{array}{l} \gamma_T(1) = \mu_T(1) \\ \gamma_T(2) = \mu_T(2) \\ \gamma_T(3) = \mu_T(3) \end{array} \Rightarrow \text{Diagonalizable}$$

$$\cancel{E_1}\quad N(\underbrace{A - 1 I_3}_{[T - 1 I_V]_\beta}) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

$$\Rightarrow \quad E_1 = N(T - 1 I_V) = \{ a1 : a \in \mathbb{R} \} \subseteq P_2(\mathbb{R})$$

Similarly, $N(A - 2I_3) = N\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$

$$E_2 = \{ a(1+x) : a \in \mathbb{R} \} \subset P_2(\mathbb{R})$$

$$E_3 = \{ a(\overbrace{1 + 2x + x^2}^{(1+x)^2}) : a \in \mathbb{R} \}$$

$\beta = \{1, 1+x, (1+x)^2\}$ is a basis of eigenvectors for V.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "{[T]_\beta}"] \arrow[d, leftrightarrow] & \mathbb{R}^3 \arrow[d, leftrightarrow] \\ P_2(\mathbb{R}) \arrow[r, "T"] & P_2(\mathbb{R}) \end{tikzcd}$$

Example: For $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

$f_A(t) = -(t-4)(t-3)^2$ splits over $\mathbb{R}$.

$$\gamma_T(4) = \mu_T(4) = 1$$

But $\operatorname{rank}(\overbrace{A - 3I}^{B}) = \operatorname{rank}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2$

$$(\underbrace{\operatorname{Rank}(B)}_{2} + \underbrace{\operatorname{Nullity}(B)}_{1} = 3)$$

$$\gamma_A(3) = 1 \neq \mu_A(3) = 2$$

$\therefore$ T is not diagonalizable.

Example: Consider $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by:

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

Let $\beta = \{1, x, x^2\}$.

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow f_T(t) = -(t-1)^2(t-2)$$

splits over $\mathbb{R}$.

and the eigenvalues of T are 1 and 2.

$\therefore \quad \gamma_T(2) = \mu_T(2) = 1$.

$$\text{Rank}([T]_\beta - I) = \text{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \Rightarrow \gamma_T(1) = 2 = \mu_T(1)$$

$\therefore$ T is diagonalizable.

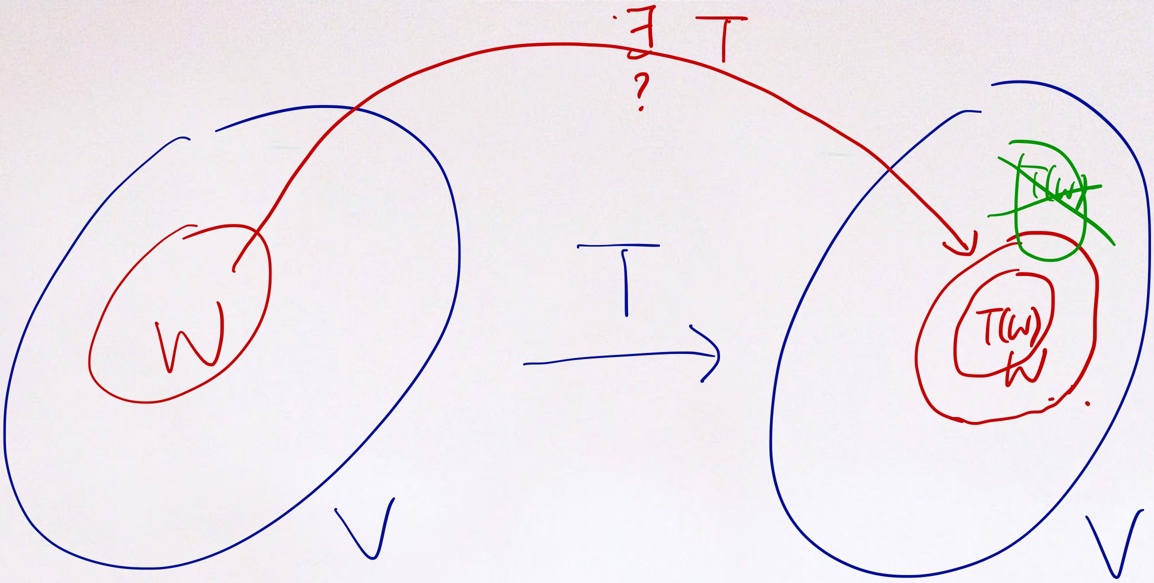
For $[T]_\beta$, the eigenspaces:

$$E_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 + x_3 = 0 \right\} = \operatorname{span} \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}}_{\text{Basis}}$$

$$E_2 = \operatorname{span} \Big\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis}} \Big\}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of eigenvectors (of $[T]_\beta$) for $\mathbb{R}^3$.

$\therefore \{1, x - x^2, 1 + x^2\}$ is a basis of eigenvectors (of T) for $P_2(\mathbb{R})$.



$$T(W) = \{ T(\vec{x}) : \vec{x} \in W \}$$

Definition: Let T be a linear operator on a vector space V.

A subspace $W \subset V$ is called T-invariant if $T(W) \subseteq W$.

That is, $T(\vec{w}) \in W$ for $\forall \vec{w} \in W$.

Example: If T is a linear operator on V, then:

$\{\vec{0}\}$ is T-invariant

V is " "

$R(T)$ " "

$N(T)$ " "

E_λ " "

eigenvalue

$(\underset{T(\vec{v})}{\vec{w}} \in R(T)$, then: $\underbrace{T(T(\vec{v}))}_{\in R(T)})$

$(\vec{v} \in E_\lambda,\ \underbrace{T(\vec{v}) = \lambda\vec{v}}_{\in E_\lambda})$

- For $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a,b,c) = (a+b,\ b+c,\ 0)$

then x-y plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ is T-invariant

x-axis $\{(x,0,0) : x \in \mathbb{R}\}$ is T-invariant

z-axis $\{(0,0,x) : x \in \mathbb{R}\}$ is NOT T-invariant.

$$T((0,0,\underset{\neq 0}{x})) = (0,\ \underset{\neq 0}{x},\ 0) \notin \text{z-axis}$$

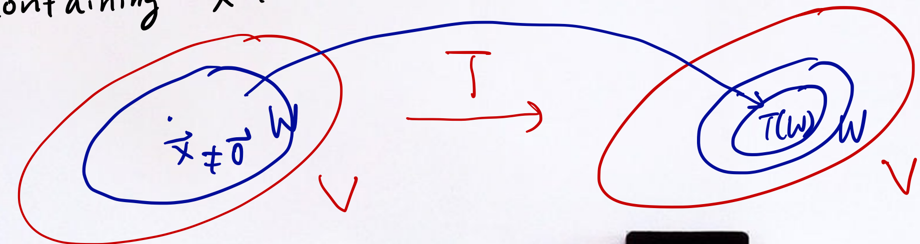
Def: Given a linear operator T on a vector space V, and a non-zero $\vec{x} \in V$, the subspace

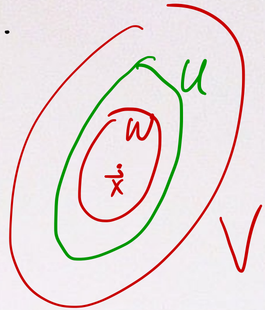
$$W := \operatorname{span}\left(\{ T^k(\vec{x}) : k \in \mathbb{N} \}\right) \overset{\text{def}}{=} \operatorname{span}\left(\{ \vec{x}, T(\vec{x}), T^2(\vec{x}), \dots, T^k(\vec{x}), \dots \}\right)$$

$$\left(T^k \overset{\text{def}}{=} \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}} \right)$$

is called **T-cyclic subspace** of V generated by $\vec{x}$.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.





<u>Proof:</u> For any $\vec{w} \in W$, $\exists\, a_0, \dots, a_k \in F$ s.t.

$$\vec{w} = \sum_{i=0}^{k} a_i T^{i}(\vec{x})$$

Then: $T(\vec{w}) = \sum_{i=0}^{k} a_i T^{i+1}(\vec{x}) \in W$.

$\therefore$ W is T-invariant.

If $U \subset V$ is a T-invariant subspace containing $\vec{x}$.

then: it also contains $\underset{\in U}{T(\vec{x})}$ and $\underset{\in U}{T^{k}(\vec{x})}$ by induction.

$\therefore$ $U \supset W$