MATH2040A/B Linear Algebra II

Final Examination

Please show all your steps, unless otherwise stated. Answer all **TEN** questions (**Total: 200 points**). Your submitted solution will be checked carefully to avoid plagiarism. Discussions amongst classmates are strictly prohibited.

1. (**10pts**) Let $V = P_2(\mathbb{C})$ be the vector space of polynomials of degree at most 2 with complex coefficients, equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\, dt.$$

Find the adjoint T^* of the linear operator $T : V \to V$ defined by

$$T(f) = if' + 2f.$$

2. (**20pts**) Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

   (a) Write A^n as a linear combination of I and A by using the Cayley-Hamilton theorem for the matrix A, and further use the derived formula to compute the exponential matrix $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$.

   (b) Determine whether A is diagonalizable or not. If yes, find an invertible Q and a diagonal matrix D such that $A = QDQ^{-1}$. Use the formula to further recompute the exponential matrix e^A, and check if the result is the same in (a).

   (c) For an arbitrary real matrix B that can be diagonalizable, give a sufficient condition such that

$$\log B = \log(I + B - I) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(B - I)^n}{n}$$

   makes sense.

3. (**20pts**) Let T be a linear operator on a finite-dimensional vector space V such that $T^2 = T$.

   (a) Show that the only possible eigenvalues of T are 0 and 1, and that $N(T)$ and $R(T)$ are the only possible eigenspaces.

   (b) Show that T is diagonalizable.

4. (**10pts**) Let V be a finite-dimensional inner product space over the real field. Assume that the linear operator $T : V \to V$ is self-adjoint and the matrix representation of T^2 in the standard basis has trace zero. Prove that $T = T_0$ is a zero transformation.

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5. (**10pts**) Let T be a linear operator on a n-dimensional vector space V over a field F. Prove that if T is invertible, then there is a polynomial $f \in P(F)$ of degree $n-1$ such that $T^{-1} = f(T)$.

6. (**20pts**) Let V be a finite-dimensional vector space over the complex field with $n = \dim(V) \geq 2$ and let $\beta = \{e_1, \cdots, e_n\}$ be a basis for V. Assume that $T : V \to V$ is a linear operator satisfying

$$T(e_i) = e_{i+1}, i = 1, \cdots, n-1; \quad T(e_n) = e_1.$$

   (a) Show that T has 1 as an eigenvalue. Find an eigenvector associated with eigenvalue 1 and show that it is unique up to scaling.

   (b) Is T diagnolizable? Justify your answer. (You may use the fact that $t = e^{\frac{2\pi ij}{n}}$ satisfies $t^n = 1$ for $i = \sqrt{-1}$ and $j = 0, 1, ..., n-1$)

7. (**20pts**) Consider a normal complex $n \times n$ matrix $A \in M_{n\times n}(\mathbb{C})$. Suppose A is positive semi-definite (that is, $\mathbf{x}^* A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^n$) and the rank of A is equal to p. Discuss whether you can find an orthogonal subset of column vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\} \subset \mathbb{C}^n$ such that:

$$A = \mathbf{v}_1 \mathbf{v}_1^* + ... + \mathbf{v}_p \mathbf{v}_p^*$$

8. (**25pts**) Let $T : V \to V$ be a normal linear operator on a n-dimensional complex inner product space V. Suppose T has k distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$.

   (a) Consider $U = g(T)$ for some non-zero polynomial g. Suppose the range $U(V)$ of U is $\bigoplus_{i=1}^{l} E_i$, where $l < k$ and E_i is the eigenspace of T associated to λ_i. What can you say about the degree of the polynomial g? Please explain your answer with details.

   (b) Suppose $T : P_3(\mathbb{C}) \to P_3(\mathbb{C})$ such that

$$T(a + bx + cx^2 + dx^3) = (4a - 2b) + (4b - 2a)x + 4cx^2 + 4dx^3.$$

   Find a non-zero polynomial g such that the range of $g(T)$ is equal to E_1, where E_1 is the eigenspace associated to the smallest eigenvalue in modulus of T.

9. (**25pts**) (**Challenging**) Let V be a finite-dimensional inner product space with an orthonormal basis $\{v_1, \cdots, v_n\}$. Assume that $u_1, \cdots, u_n$ are vectors in V such that

$$\sum_{j=1}^{n} \|u_j\|^2 < 1$$

where $\|\cdot\|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$. Show that

$$\{v_1 + u_1, \cdots, v_n + u_n\}$$

is a basis for V.

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10. (**40pts**) (**Challenging**) Let $T : V \to V$ be a self-adjoint linear operator on a n-dimensional inner product space V over the field $F = \mathbb{C}$. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be its eigenvalues arranged in the ascending order and counted with multiplicity.

   (a) Explain why $\langle x, T(x) \rangle$ is a real number for all $x \in V$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. Please prove your answer with details.

   (b) Consider:

   $$m(W) = \min\{\langle x, T(x) \rangle \mid x \in W \text{ and } \langle x, x \rangle = 1\} \text{ and}$$
   $$M(W) = \max\{\langle x, T(x) \rangle \mid x \in W \text{ and } \langle x, x \rangle = 1\},$$

   where W is a subspace of V. What can you say about the relationship amongst $m(V)$, $M(V)$ and the eigenvalues of T? Please prove your answer with details.

   (c) Now, consider:

   $$R = \min\{M(W) \mid \dim(W) = k\} \text{ and } r = \max\{m(W) \mid \dim(W) = n - k + 1\}.$$

   What can you say about the relationship between R and the eigenvalues of T. Similarly, can you say about the relationship between r and the eigenvalues of T? Please prove your answers with details.

   (d) Suppose $U : V \to V$ is another self-adjoint linear operator with eigenvalues $\mu_1 \leq \mu_2 \leq ... \leq \mu_n$ (counted with multiplicity). Assume the eigenvalues of $T+U$ are given by $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n$ (counted with multiplicity). Let $1 \leq i, j, k \leq n$. If $i + j = n + k$, prove that $\gamma_k \leq \lambda_i + \mu_j$.

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