MMAT5390 Mathematical Image Processing
Practice Midterm Examination

Abstract

This practice exam gathers questions from each chapter(with or without minor changes), authors' creation and past exams that are more **likely** to resemble questions in the Midterm Exam. Yet, it does not mean that any of the following will appear. It is up to you to decide how important this practice exam is.

1. Suppose $H \in M_{4\times 4}$ is applied to a 2×2 image. Let

$$H = \begin{pmatrix} 3 & 3 & 6 & 0 \\ 0 & 3 & 3 & 6 \\ 6 & 0 & 3 & 3 \\ 3 & 6 & 0 & 3 \end{pmatrix}$$

Is H shift-invariant with h_s being 2-periodic? Please explain your answer with details.

2. Let $f = (f(m,n))_{-2\le m,n\le 2}$ be a 5×5 image. Consider a filter $H = (h(m,n))_{-2\le m,n\le 2}$, which is another 5×5 image. Note that the indices are taken from -2 to 2. Suppose $H = (a_1, a_2, a_3, a_4, a_5)^T(b_1, b_2, b_3, b_4, b_5)$.

   (a) Define the discrete convolution $H * f$.

   (b) Show that $H * f = H_1 * (H_2 * f)$, where

$$H_1 = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & 0 & 0 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Hence, $H * f$ can be computed by a sequence of 1D convolutions.

3. Let H be a $(2N+1) \times (2N+1)$ matrix. Let $\mathcal{I}$ be the collection of $(2N+1) \times (2N+1)$ images. Assume the indices are taken from $-N$ to N. Define: $\mathcal{O}(H) : \mathcal{I} \to \mathcal{I}$ by:

$$\mathcal{O}(I) = I * H$$

where $I * H$ refers to the discrete convolution.

   (a) Give the definition of discrete convolution. Argue that $\mathcal{O}$ is a linear operator and shift-invariant.

   (b) Show that $I * (H_1 * H_2) = (I * H_1) * H_2$, where H_1 and H_2 are $(2N+1) \times (2N+1)$ matrices.

   (c) Show that $I * H = H * I$.

4. Let $H_1 = \begin{pmatrix} 4 & 5 & 7 & 3 \\ 3 & 4 & 5 & 7 \\ 7 & 3 & 4 & 5 \\ 5 & 7 & 3 & 4 \end{pmatrix}$ and $H_2 = \begin{pmatrix} 9 & 9 & 18 & 9 & 9 & 18 & 18 & 18 & 36 \\ 18 & 9 & 9 & 18 & 9 & 9 & 36 & 18 & 18 \\ 9 & 18 & 9 & 9 & 18 & 9 & 18 & 36 & 18 \\ 18 & 18 & 36 & 9 & 9 & 18 & 9 & 9 & 18 \\ 36 & 18 & 18 & 18 & 9 & 9 & 18 & 9 & 9 \\ 18 & 36 & 18 & 9 & 18 & 9 & 9 & 18 & 9 \\ 9 & 9 & 18 & 18 & 18 & 36 & 9 & 9 & 18 \\ 18 & 9 & 9 & 36 & 18 & 18 & 18 & 9 & 9 \\ 9 & 18 & 9 & 18 & 36 & 18 & 9 & 18 & 9 \end{pmatrix}$.

   Discuss whether H_1 and H_2 represent shift-invariant linear transformations (with h_s being N-periodic in both arguments) on $N \times N$ square images. Please explain your answer with details.

5. (Optional) Prove that a real symmetric matrix is always diagonalizable and has real eigenvalues.

6. (Optional) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a square matrix A. Prove that $tr(A^p) = \sum_i \lambda_i^p$, where $tr(\cdot)$ denotes the trace operator.

7. Let $f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$.

   (a) Compute an SVD of f.

(b) Express f as a linear combination of its elementary images.

8. Let $f = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$.

   (a) Compute the Haar transform f_{Haar} of f.

   (b) Suppose there is only enough capacity to store 10 pixel values of f_{Haar}. Choose 10 entries to keep such that the reconstucted image differs as little as possible in Frobenius norm with the original image, and compute the reconstructed image.

9. Let $f = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -2 & 4 & 1 \\ -2 & -1 & -1 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}$.

   (a) Compute the Walsh transform f_{Walsh} of f.

   (b) Suppose f_{Walsh} is mistakenly stored as an integer-valued matrix, with each entry rounded away from 0. Compute the reconstructed image from the modified matrix.

10. Let $\mathcal{W} := \{W_m : m \in \mathbb{N} \cup \{0\}\}$ be the sequence of Walsh functions.

    (a) (Unit) Prove that $\int_{\mathbb{R}} [W_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$.
    (Hence $W_m \in L^2(\mathbb{R})$ and $\|W_m\| = 1$.)

    (b) (Orthogonality) Let $P(m)$ be the proposition that $\{W_0, \ldots, W_m\}$ is orthogonal in $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$. $P(0)$ is vacuously true. We aim to establish:

    $$P(m) \text{ is true} \implies P(2m+1) \text{ is true} \qquad (\star)$$

    whenever $m \in \mathbb{N} \cup \{0\}$. Also note that $P(m)$ implies $P(m-1)$ for any $m \in \mathbb{N} \setminus \{0\}$ (since the latter concerns the orthogonality of a subset of the set concerned by the former), so by induction we establish $P(m)$ to be true for all $m \in \mathbb{N} \setminus \{0\}$ once $(\star)$ holds.
    Suppose $P(k)$ holds for some $k \in \mathbb{N} \cup \{0\}$.
    Let $m_1, m_2 \in \mathbb{Z} \cap [0, 2k+1]$ such that $m_1 < m_2$. Then $m_1 = 2j_1 + q_1$ and $m_2 = 2j_2 + q_2$ for some $j_1, j_2 \in \mathbb{N} \cup \{0\}$ and $q_1, q_2 \in \{0, 1\}$.

    i. Suppose $j_1 = j_2$. Prove that $\langle W_{m_1}, W_{m_2} \rangle = 0$.
    Hint. In this case $q_1 = 0$, $q_2 = 1$.

    ii. Suppose $j_1 < j_2$. Prove that $\langle W_{m_1}, W_{m_2} \rangle = 0$.
    Hint. $j_2 \leq n$. Make good use of the induction hypothesis.

    The above establishes that $\mathcal{W}$ is orthonormal in $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$.

11. Let $f = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}$.

    (a) Compute the discrete Fourier transform $\hat{f}$ of f.

    (b) Compute the image reconstructed from $\hat{f}$ using only the 3 frequencies closest to 0.

12. Verify the following properties of the discrete Fourier transform (i) directly; (ii) via the inverse transform.

    (a) Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended. Then $\widehat{f * g} = MN \hat{f} \odot \hat{g}$, where $\hat{f} \odot \hat{g}(m, n) = \hat{f}(m, n)\hat{g}(m, n)$.

    (b) Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended. Then $\widehat{f \odot g} = \hat{f} * \hat{g}$, where $f \odot g(k, l) = f(k, l)g(k, l)$.

    (c) Let $f \in M_{N \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k, l) = f(l, -k)$. Then $\hat{\tilde{f}} = \tilde{\hat{f}}$.

    (d) Let $f \in M_{M \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k, l) = f(k - k_0, l - l_0)$ for some $k_0, l_0 \in \mathbb{Z}$. Then $\hat{\tilde{f}} = e^{-2\pi j(\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f}$.

(e) Let $f \in M_{M\times N}(\mathbb{R})$ be periodically extended, and let $\tilde{\hat{f}}(m,n) = \hat{f}(m-m_0, n-n_0)$ for some $m_0, n_0 \in \mathbb{Z}$. Then $\tilde{\hat{f}} = DFT(e^{2\pi j(\frac{km_0}{M}+\frac{ln_0}{N})} f)$.

13. Compute the degradation functions in the frequency domain that correspond to the following $M \times N$ convolution kernels h, i.e. find $H \in M_{M\times N}(\mathbb{C})$ such that

$$DFT(h * f)(u,v) = H(u,v) DFT(f)(u,v)$$

for any periodically extended $f \in M_{M\times N}(\mathbb{R})$:

(a) Assuming integer k satisfies $k \leq \min\{\frac{M}{2}, \frac{N}{2}\}$,

$$h_1(x,y) = \begin{cases} \frac{1}{(2k+1)^2} & \text{if } dist(x, M\mathbb{Z}) \leq k \text{ and } dist(y, N\mathbb{Z}) \leq k, \\ 0 & \text{otherwise;} \end{cases}$$

(b) Letting $r > 1$,

$$h_2(x,y) = \begin{cases} \frac{r}{r+4} & \text{if } D(x,y) = 0, \\ \frac{1}{r+4} & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(c)

$$h_3(x,y) = \begin{cases} \frac{1}{4} & \text{if } D(x,y) = 0, \\ \frac{1}{8} & \text{if } D(x,y) = 1, \\ \frac{1}{16} & \text{if } D(x,y) = 2, \\ 0 & \text{otherwise;} \end{cases}$$

(d)

$$h_4(x,y) = \begin{cases} -4 & \text{if } D(x,y) = 0, \\ 1 & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(e) Letting $a, b \in \mathbb{Z}$ and $T \in \mathbb{N} \setminus \{0\}$ such that $|a|(T-1) < M$ and $|b|(T-1) < N$,

$$h_5(x,y) = \begin{cases} \frac{1}{T} & \text{if } (x,y) \in \{(at, bt) : t = 0, 1, \cdots, T-1\} \\ 0 & \text{otherwise.} \end{cases}$$

14. Prove that for any $f \in M_{M\times N}(\mathbb{C})$,

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |DFT(f)(m,n)|^2 = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} |f(k,l)|^2 .$$

15. Recall that the 0-th Haar function is

$$H_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the other Haar functions are defined by

$$H_{2^p+n}(t) = \begin{cases} 2^{\frac{p}{2}} & \text{if } \frac{n}{2^p} \leq t < \frac{n+0.5}{2^p} \\ -2^{\frac{p}{2}} & \text{if } \frac{n+0.5}{2^p} \leq t < \frac{n+1}{2^p} \\ 0 & \text{otherwise} \end{cases}$$

for $p = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots, 2^p - 1$. Denote the inverse Haar transform by iHT.
Let $g = (g(k,l))_{0\leq k,l\leq 2^N-1}$ be a $2^N \times 2^N$ image. Suppose the Haar transform of g is given by $\tilde{g} = (\tilde{g}(m,n))_{0\leq m,n\leq 2^N-1}$ where $\tilde{g}(m,n) = e^{K-m^2-n^2}$ for some positive constant $K > 0$.
Let $h^* = (h^*(m,n))_{0\leq m,n\leq 2^N-1}$ such that

$$h^*(m,n) = \begin{cases} \tilde{g}(m,n) & \text{if } m^2+n^2 < K, \\ 0 & \text{if } m^2+n^2 \geq K. \end{cases}$$

Prove that h^* minimizes

$$E(h) = \|h\|_0 + \|iHT(h) - g\|_F^2$$

over $h \in \mathcal{S}(\tilde{g})$, where $\|\cdot\|_0$ is called the 0-norm and counts the number of nonzero entries, and for $A \in M_{p\times q}$,

$$\mathcal{S}(A) = \{B \in M_{p\times q} : B(i,j) = 0 \text{ or } B(i,j) = A(i,j) \text{ for each } 1 \leq i \leq p,\ 1 \leq j \leq q\}.$$

(For instance,

$$\mathcal{S}(\begin{pmatrix}1\\2\end{pmatrix}) = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}.)$$