MMAT5390 Mathematical Image Processing
Midterm Examination

You have to answer all five questions. **Please show your steps** unless otherwise stated.

1. This question is about Haar transformation.

   (a) Compute the Haar transform $\tilde{f}$ of the following 4×4 image

   $$f = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & s & 0 \end{pmatrix},$$

   where r and s are two non-zero real numbers.

   (b) Following (a), show that f can be written as a linear combination of exactly four elementary images under the Haar transformation if and only if $r = s$ or $r = -s$. Please explain your answer in details.

   (c) Following (a) and (b), if $r = -s$, write f as a linear combination of four elementary images under the Haar transformation.

2. This question is about image decomposition using singular value decomposition (SVD). Consider:

   $$A = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   where a, b and c are positive real numbers and $a \geq b \geq c$.

   (a) Compute the SVD of A. Please show all your steps. (**Hint:** Compute $A^T A$ and find the eigenvalues of $A^T A$.)

   (b) Write A as a linear combination of eigen-images.

   (c) Suppose A is degraded during the transmission process to get the corrupted image $\tilde{A}$. Suppose $\tilde{A}$ is the same as A except for the 4-th row 1-st column entry. In particular, $\tilde{A}(4,1) = \epsilon$. Using (a) and (b), find the SVD of $\tilde{A}$. Please explain your answer with details.

3. Let $H = \begin{pmatrix} r & 2r & u & 2u \\ 3r & r & 3v & v \\ 3 & 6 & s & 2s \\ 9 & 3 & 3s & s \end{pmatrix}$ be the transformation matrix corresponding to a point spread function $h = h(x, \alpha, y, \beta)$, where r, s, u, v are all non-zero real numbers. Prove that h is both separable and shift-invariant if and only if $r = s$ and $u = v$ (here, we do not assume h to be periodic in each variables). Please explain your answer with details.

PLEASE TURN TO THE NEXT PAGE

4. This question studies how a blurry image is degraded from its original image. Let $I = (I(m,n))_{0\leq m,n\leq N-1}$ be a $N \times N$ image, which is periodically extended. Suppose $\tilde{I}$ be a blurry image given by:

$$\tilde{I}(x,y) = I(x,y) + \sum_{k=1}^{L} r^k I(x-k, y+k) \text{ for } 0 \leq x,y \leq N-1,$$

where $0 < r < 1$. Denote the discrete Fourier transforms of I and $\tilde{I}$ by $DFT(I)$ and $DFT(\tilde{I})$ respectively. Prove that $DFT(\tilde{I})(u,v) = H(u,v)DFT(I)(u,v)$ for some H, where $0 \leq u,v \leq N-1$. What is H in terms of N, u, v, r and L? Please derive your answer from the definiton of DFT and show all your steps clearly (including how the changes of variables are applied, indices are shifted and so on). Missing details will lead to mark deduction.

5. Let $g = (g(m,n))_{0\leq m\leq M-1, 0\leq n\leq N-1} \in M_{M\times N}(\mathbb{R})$ be a $M \times N$ image. Assume g is periodically extended. Suppose $\tilde{g}$ is obtained by translating and rotating g. More specifically,

$$\tilde{g}(m,n) = g(3-m, 5-n), \text{ where } 4 \leq m \leq M+3 \text{ and } -4 \leq n \leq N-5.$$

Express $\hat{\tilde{g}}$ in terms of $\hat{g}$, where $\hat{\tilde{g}}$ is the DFT of $\tilde{g}$ and $\hat{g}$ is the DFT of g. Please derive your answer from the definiton of DFT and show all your steps clearly (including how the changes of variables are applied, indices are shifted and so on). Missing details will lead to mark deduction.

END OF PAPER