# MMAT5390: Mathematical Image Processing

## Assignment 1

Due: Fe8bruary 2022

Please give reasons in your solutions.

1. (a) For the following transformation matrices, determine if their corresponding point spread functions (PSF) are separable and/or shift-invariant (with periodical extension assumption). If not, give your reason.

   i. $H = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix}$;

   ii. $H = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 & 6 & 0 & 0 & 0 \\ 8 & 10 & 0 & 8 & 10 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 3 & 6 & 9 \\ 4 & 5 & 0 & 0 & 0 & 0 & 12 & 15 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 12 & 1 & 2 & 3 \\ 0 & 0 & 0 & 16 & 20 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}$;

   (b) Consider the following point spread functions, determine whether they are separable and/or shift-invariant. If not, provide your reason.

   i. $h(x, \alpha, y, \beta) = (\alpha - x)(\beta - y) + (\alpha - x)^2$;

   ii. $h(x, \alpha, y, \beta) = \frac{\alpha e^{-x^2}}{\beta^2 + y^2 + 5}$

2. (a) Let $H$ be the transformation matrix of shift-invariant transformation on $N \times N$ images. Assume that the images are periodically extended. Show that $H$ is *block-circulant*.

   (b) Here we consider a general case of (a). Let $H$ be the transformation matrix of shift-invariant transformation on $N \times N$ images. We do not assume that the images are periodically extended. Show that $H$ is *block-Toeplitz*.

   *Remark*: a Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant. For example, the following matrix $T$ is Toeplitz

$$T = \begin{pmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \\ g & f & e & a \end{pmatrix}$$

   A block Toeplitz matrix is another special kind of block matrix, which contains blocks that are repeated down the diagonals of the matrix, as a Toeplitz matrix has elements repeated down the diagonal. The individual block matrix elements, $A_{ij}$, must also be a Toeplitz matrix. A block-Toeplitz matrix has the form

   $A = \begin{pmatrix} A_0 & A_{-1} & \cdots & \\ A_1 & A_0 & A_{-1} & \cdots \\ A_2 & A_1 & A_0 & \cdots \\ \cdots & \cdots & & \end{pmatrix}$, where matrices $A_0, A_1, \cdots$ are Toeplitz.

3. Let $f = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 4 \\ 0 & 6 \end{pmatrix}$, assume $f$ and $g$ are periodically extended, $*$ denotes convolution operation.

(a) Compute $f * g$ and $g * f$.

(b) Show that $f * g = g * f$ when f and g are two $m \times n$ images with periodical extension.

4. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(a) Compute an SVD of $A$. Please show all your steps.

(b) Write $A$ as a linear combination of its elementary images from SVD. Please show all your steps.

5. (a) Suppose the transformation matrix $H = \begin{pmatrix} 2 & 0 & 8 & 0 \\ 1 & 2 & 4 & 8 \\ 6 & 0 & 4 & 0 \\ 3 & 6 & 2 & 4 \end{pmatrix}$, is this transformation separable? If yes, find out $g_1$ and $g_2$ such that $h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta)$;

(b) Consider the general case, if the PSF of a linear image transformation is given by $h(x, \alpha, y, \beta) = h_c(x, \alpha) h_r(y, \beta)$, show that the transformation matrix $H = h_r^T \otimes h_c^T$.