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二〇二一至二〇二二年度下學期科目考試
Course Examination 2nd Term, 2021-22

科目編號及名稱 Course Code & Title : **MMAT5390 Mathematical Image Processing**

時間 Time allowed : ________ 小時 hours ________ 分鐘 minutes

學號 Student I.D. No : ________ 座號 Seat No. : ________

**Please show all your steps**, unless otherwise stated. Answer all **five** questions. The total score is 100. Your submitted solution will be checked carefully to avoid plagiarism. Discussions amongst classmates are strictly prohibited.

1. Consider a $4 \times 4$ periodically extended image $I = (I(k,l))_{0 \le k,l \le 3}$ given by:

$$I = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

where $a, b \ge 0$.

The Wiener filter $T = (T(u,v))_{0 \le u,v \le 3}$ for image deblurring is defined by

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

where $K$ is a constant, $H(u,v) = DFT(h)(u,v)$ with $h$ being a bluring convolution kernel:

$$h = \begin{pmatrix} 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\hat{I}(u,v) = T(u,v)DFT(I)(u,v)$.

Suppose

$$\begin{cases} DFT(I)(0,0) = \frac{1}{2}, \\ DFT(I)(2,0) = \frac{1}{4}, \\ \hat{I}(2,0) = \frac{4}{33}. \end{cases}$$

(a) Compute $DFT(h)$ and $DFT(I)$;

(b) Find $a, b$ and $K$, and derive $\hat{I}$.

2. Consider a periodically extended $4 \times 4$ image $I = (I(x,y))_{0 \le x,y \le 3}$ given by:

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & a & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given that the discrete Laplacian $\Delta I$ of $I$ is given by the formula:

$\Delta I(x,y) = -4I(x,y) + I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1)$ for $0 \le x, y \le 3$.

We perform the Laplacian masking on $I$ to get a sharpen image $I_{sharp}$. Suppose $I_{sharp}$ is given by

$$I_{sharp} = \begin{pmatrix} 0 & 0 & 5 & -3 \\ -1 & -5 & -1 & 5 \\ 5 & 0 & 0 & -3 \\ -2 & 0 & -2 & 5 \end{pmatrix}.$$

Find $a$ and $b$. (**Hint:** You may want to use the formula of Laplacian masking in the spatial domain directly: $I_{sharp} = I - \Delta I$.)

3. This question is related to the energy minimization approach to solve the image denoising problem in Class Note 12. Given a noisy image $I : D \to \mathbb{R}$, we consider the following image denoising model to restore the original clean image $f : D \to \mathbb{R}$ that minimizes:

$$E(f) = \frac{1}{2} \int_D (f(x,y) - I(x,y))^2 \, dxdy + \int_D K(x,y) \sqrt{|\nabla f(x,y)|^2 + \epsilon} \, dxdy$$

for some positive function $K : D \to \mathbb{R}$ and small parameter $\epsilon > 0$. Suppose $f$ minimizes $E(f)$. Show that $f$ must satisfy the following partial differenital equation in $D$:

$$-\nabla \cdot \left( K(x,y) \frac{\nabla f(x,y)}{\sqrt{|\nabla f(x,y)|^2 + \epsilon}} \right) + f(x,y) = I(x,y) \text{ for } (x,y) \in D$$

Please show your answers with all details including all derivations. Missing details will lead to mark deductions.

4. Consider the following curve evolution model for image segmentation. Let $\gamma : [0, 2\pi] \to D$ be a closed curve in the image domain $D$. We proceed to find $\gamma$ that minimizes:

$$E_{snake}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

where $\alpha$ and $\beta$ are fixed positive parameters and $V(x, y) = x^2 + y^2$ is the edge detector.

(a) Derive the gradient descent iterative scheme to minimize $E_{snake}$ in the continuous setting. That is, derive an iterative scheme to find a sequence of contours $\{\gamma^0, \gamma^1, ..., \gamma^n, ...\}$ such that:

$$\frac{\gamma^{n+1} - \gamma^n}{\Delta t} = G(\gamma^n),$$

for some scheme $G$. What is $G$? (Do not leave your answers in term of $V$) Please explain your answer with details. Missing details will lead to mark deductions.

(b) Let $\gamma^0(s) = (\cos s, \sin s)$. Prove that $\gamma_n$ converges to the origin if $\Delta t$ is small enough. In other words, $\lim_{n\to\infty} \gamma_n(s) = (0, 0)$ for all $s \in [0, 2\pi]$. Please show your answer with all details.

5. This question is similar to the constrained least square filtering in Class Note 10. Consider a noisy $N \times N$ image $g \in M_{N\times N}(\mathbb{R})$. Let $\mathcal{S}$ be the stacking operator. That is, $\mathcal{S}(g)$ is the vectorized image of $g$. Suppose $f$ is a $N \times N$ image that minimizes:

$$E(f) = ||f_x||_F^2 + ||f_y||_F^2,$$

subject to the constraints that $||g - h * f||_F^2 = \epsilon$, where $*$ denotes the convolution, $||\cdot||_F$ denotes the Frobenius norm and $h$ is a $N \times N$ convolution kernel; $f_x$ and $f_y$ are two $N \times N$ images given by:

$$f_x(u, v) = f(u + 1, v) - f(u, v) \text{ and } f_y(u, v) = f(u, v + 1) - f(u, v),$$

where $0 \le u, v \le N - 1$ (assuming that $f$ is periodically extended).

(a) Prove that $(\lambda D^T D + L_1^T L_1 + L_2^T L_2)\mathcal{S}(f) = \lambda D^T \mathcal{S}(g)$ for some $\lambda \in \mathbb{R}$, where $D$, $L_1$ and $L_2$ are $N^2 \times N^2$ matrices satisfying $\mathcal{S}(h * f) = D\mathcal{S}(f)$, $\mathcal{S}(f_x) = L_1\mathcal{S}(f)$ and $\mathcal{S}(f_y) = L_2\mathcal{S}(f)$. Please show your answer with all details. Missing details will lead to mark deductions.

(b) We will now consider the DFT of $f$ and $g$, which are denoted by $DFT(f)$ and $DFT(g)$ respectively.

i. Write $f_x = p_1 * f$ and $f_y = p_2 * f$, where $*$ denotes the convolution, $p_1$ and $p_2$ are two $N \times N$ convolution kernels. What are $p_1$ and $p_2$?

ii. Prove that

$$DFT(f)(u, v) = \frac{1}{N^2}\left(\frac{\lambda\overline{H(u, v)}}{\lambda|H(u, v)|^2 + |P_1(u, v)|^2 + |P_2(u, v)|^2}\right) DFT(g)(u, v),$$

where $H$, $P_1$ and $P_2$ are the DFT of $h$, $p_1$ and $p_2$ respectively. Please show your answer with all details. Missing details will lead to mark deductions.

**END OF PAPER**