

# MMAT5390: Mathematical Image Processing

## Solutions to Chapter 5 Exercises

1. Recall that the result  $g_i$  of the  $3 \times 3$  mean filter applied on  $f \in M_{M \times N}(\mathbb{R})$  is given by:

$$g_i(x, y) = \frac{1}{9} \sum_{k=-1}^1 \sum_{l=-1}^1 f(x+k, y+l);$$

the result  $g_{ii}$  of the  $3 \times 3$  median filter applied on  $f \in M_{M \times N}(\mathbb{R})$  is given by:

$$g_{ii}(x, y) = \text{median}\{f(x+k, y+l) : k, l \in \{-1, 0, 1\}\};$$

the result  $g_{iii}$  of the convolution filter  $\frac{1}{8} \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$  applied on  $f \in M_{4 \times 4}(\mathbb{R})$  is given by:

$$g_{iii}(x, y) = \frac{1}{2}f(x, y) + \frac{1}{8}[f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)].$$

Hence the results from applying the filters on the given images are:

(a) i.  $\frac{1}{9} \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$

ii.  $\mathbf{0}$

iii.  $\frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$

(b) i.  $\frac{1}{9} \begin{pmatrix} 7 & 6 & 7 & 7 \\ 7 & 7 & 7 & 6 \\ 6 & 7 & 7 & 7 \\ 7 & 7 & 6 & 7 \end{pmatrix}$

ii.  $\mathbf{1}$

iii.  $\frac{1}{8} \begin{pmatrix} 7 & 7 & 4 & 7 \\ 4 & 7 & 7 & 7 \\ 7 & 7 & 7 & 4 \\ 7 & 4 & 7 & 7 \end{pmatrix}$

(c) i.  $\frac{1}{9} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

ii.  $\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

iii.  $\mathbf{0}$

2. (a) For any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned} h_1 * f(x, y) &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_1(k, l) f(x-k, y-l) \\ &= f(x+1, y) - f(x, y). \end{aligned}$$

Then for any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned}
& \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_1 * h_1(k, l) f(x-k, y-l) \\
&= (h_1 * h_1) * f(x, y) \\
&= h_1 * (h_1 * f)(x, y) \\
&= h_1 * f(x+1, y) - h_1 * f(x, y) \\
&= f(x+2, y) - f(x+1, y) - [f(x+1, y) - f(x, y)] \\
&= f(x+2, y) - 2f(x+1, y) + f(x, y).
\end{aligned}$$

$$\text{Hence } h_1 * h_1(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0) \text{ or } (-2, 0), \\ -2 & \text{if } (x, y) = (-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

(b) For any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned}
h_2 * f(x, y) &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_2(k, l) f(x-k, y-l) \\
&= \frac{1}{9} \sum_{k=-1}^1 \sum_{l=-1}^1 f(x-k, y-l).
\end{aligned}$$

Then for any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned}
& \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_2 * h_2(k, l) f(x-k, y-l) \\
&= (h_2 * h_2) * f(x, y) \\
&= h_2 * (h_2 * f)(x, y) \\
&= \frac{1}{9} \sum_{k=-1}^1 \sum_{l=-1}^1 h_2 * f(x-k, y-l) \\
&= \frac{1}{81} \sum_{k, k', l, l'=-1}^1 f(x-k-k', y-l-l').
\end{aligned}$$

Note that for  $k, k' \in \{-1, 0, 1\}$ ,  
 $k+k' = -2$  if and only if  $(k, k') = (-1, -1)$ ,  
 $k+k' = -1$  if and only if  $(k, k') = (-1, 0)$  or  $(0, -1)$ ,  
 $k+k' = 0$  if and only if  $(k, k') = (-1, 1)$  or  $(0, 0)$  or  $(1, -1)$ ,  
 $k+k' = 1$  if and only if  $(k, k') = (0, 1)$  or  $(1, 0)$ ,  
 $k+k' = 2$  if and only if  $(k, k') = (1, 1)$ .

The same goes for  $l+l'$ . Hence

$$h_2 * h_2(x, y) = \begin{cases} \frac{1}{81}(3-|x|)(3-|y|) & \text{if } x, y \in \{-2, -1, 0, 1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

(c) For any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned}
h_3 * f(x, y) &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_3(k, l) f(x-k, y-l) \\
&= \frac{1}{2} f(x, y) + \frac{1}{4} [f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)].
\end{aligned}$$

Then for any  $f \in M_{N \times N}(\mathbb{R})$  and  $x, y \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned}
& \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} h_3 * h_3(k, l) f(x-k, y-l) \\
&= (h_3 * h_3) * f(x, y) \\
&= h_3 * (h_3 * f)(x, y) \\
&= \frac{1}{2} h_3 * f(x, y) + \frac{1}{4} [h_3 * f(x+1, y) + h_3 * f(x-1, y) + h_3 * f(x, y+1) + h_3 * f(x, y-1)] \\
&= \frac{1}{2} \left\{ \frac{1}{2} f(x, y) + \frac{1}{4} [f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)] \right\} \\
&+ \frac{1}{4} \left\{ \frac{1}{2} f(x+1, y) + \frac{1}{4} [f(x+2, y) + f(x, y) + f(x+1, y+1) + f(x+1, y-1)] \right\} \\
&+ \frac{1}{4} \left\{ \frac{1}{2} f(x-1, y) + \frac{1}{4} [f(x, y) + f(x-2, y) + f(x-1, y+1) + f(x-1, y-1)] \right\} \\
&+ \frac{1}{4} \left\{ \frac{1}{2} f(x, y+1) + \frac{1}{4} [f(x+1, y+1) + f(x-1, y+1) + f(x, y+2) + f(x, y)] \right\} \\
&+ \frac{1}{4} \left\{ \frac{1}{2} f(x, y-1) + \frac{1}{4} [f(x+1, y-1) + f(x-1, y-1) + f(x, y) + f(x, y-2)] \right\} \\
&= \frac{1}{2} f(x, y) + \frac{1}{4} [f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)] \\
&+ \frac{1}{8} [f(x+1, y+1) + f(x-1, y+1) + f(x+1, y-1) + f(x-1, y-1)] \\
&+ \frac{1}{16} [f(x+2, y) + f(x-2, y) + f(x, y+2) + f(x, y-2)].
\end{aligned}$$

Hence

$$h_3 * h_3(x, y) = \begin{cases} \frac{1}{2} & \text{if } D(x, y) = 0, \\ \frac{1}{4} & \text{if } D(x, y) = 1, \\ \frac{1}{8} & \text{if } D(x, y) = 2, \\ \frac{1}{16} & \text{if } D(x, y) = 4, \\ 0 & \text{otherwise.} \end{cases}$$

3. (a) i. Suppose  $f$  is a minimizer of  $E_1$ . Then for any  $v : \Omega \rightarrow \mathbb{R}$ ,

$$\begin{aligned}
0 &= \left. \frac{\partial}{\partial t} \right|_{t=0} E_1(f + tv) \\
&= \int_{\Omega} \left. \frac{\partial}{\partial t} \right|_{t=0} [(f + tv - g)^2 + \lambda \|\nabla(f + tv)\|_2^2] dx dy \\
&= \int_{\Omega} [2v(f - g) + \lambda \left. \frac{\partial}{\partial t} \right|_{t=0} \langle \nabla f + t\nabla v, \nabla f + t\nabla v \rangle] dx dy \\
&= \int_{\Omega} [2v(f - g) + \lambda \left. \frac{\partial}{\partial t} \right|_{t=0} (\|\nabla f\|_2^2 + 2t\langle \nabla f, \nabla v \rangle + t^2\|\nabla v\|_2^2)] dx dy \\
&= 2 \int_{\Omega} [v(f - g) + \lambda \langle \nabla f, \nabla v \rangle] dx dy \\
&= 2 \int_{\Omega} v[f - g - \lambda \nabla \cdot (\nabla f)] + 2\lambda \int_{\partial\Omega} v \langle \nabla f, \vec{n} \rangle ds. \tag{*}
\end{aligned}$$

Since  $v$  is arbitrarily chosen,  $f$  satisfies:

$$\begin{cases} -\lambda \nabla^2 f(x, y) + f(x, y) = g(x, y) & \text{if } (x, y) \in \Omega, \\ \langle \nabla f(x, y), \vec{n}(x, y) \rangle = 0 & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

ii. Suppose  $f$  satisfies the above PDE. Then for any  $h : \Omega \rightarrow \mathbb{R}$ ,

$$\begin{aligned}
E_1(h) - E_1(f) &= \int_{\Omega} [(h-g)^2 - (f-g)^2 + \lambda \|\nabla h\|_2^2 - \lambda \|\nabla f\|_2^2] dx dy \\
&= \int_{\Omega} [h^2 - 2gh + g^2 - f^2 + 2fg - g^2 + \lambda(\|\nabla h - \nabla f\|_2^2 - 2\|\nabla f\|_2^2 + 2\langle \nabla h, \nabla f \rangle)] dx dy \\
&= \int_{\Omega} [(h-f)^2 + 2hf - 2f^2 - 2gh + 2fg + \lambda(\|\nabla h - \nabla f\|_2^2 + 2\langle \nabla f, \nabla h - \nabla f \rangle)] dx dy \\
&\geq 2 \int_{\Omega} [(f-g)(h-f) + \lambda \langle \nabla f, \nabla h - \nabla f \rangle] dx dy \\
&= 2 \int_{\Omega} [(f-g)(h-f) - \lambda(h-f)\nabla^2 f] dx dy + 2 \int_{\partial\Omega} (h-f)\langle \nabla f, \vec{n} \rangle ds = 0.
\end{aligned}$$

Hence  $f$  minimizes  $E_1$ .

iii. From (\*), a descent direction of  $E_1$  at  $f$  is given by:

$$v(x, y) = \begin{cases} -f(x, y) + g(x, y) + \lambda \nabla^2 f(x, y) & \text{if } (x, y) \in \Omega, \\ -\langle \nabla f(x, y), \vec{n}(x, y) \rangle & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

Hence given  $f : \Omega \rightarrow \mathbb{R}$ , the gradient descent scheme for minimizing  $E_1$  is given by:

$$\frac{\partial f(x, y; t)}{\partial t} = \begin{cases} -f(x, y) + g(x, y) + \lambda \nabla^2 f(x, y) & \text{if } (x, y) \in \Omega, \\ -\langle \nabla f(x, y), \vec{n}(x, y) \rangle & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

Upon discretization in time and space, the explicit gradient descent scheme with time step  $\tau$  becomes:

$$\frac{f^{n+1}(x, y) - f^n(x, y)}{\tau} = \begin{cases} -f^n(x, y) + g(x, y) + \lambda \mathcal{D}_2(\mathcal{D}_1(f^n))(x, y) & \text{if } (x, y) \in \Omega, \\ -(\mathcal{D}_1 f^n(x, y), \vec{n}(x, y)) & \text{if } (x, y) \in \partial\Omega, \end{cases}$$

where the finite difference operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  approximate the differential operators  $\nabla$  and  $\nabla \cdot$ .

iv. Using the forward difference approximations for first derivatives,

$$E_{1,discrete}(f) = \sum_{(x,y) \in \Omega} \{ [f(x, y) - g(x, y)]^2 + \lambda [f(x+1, y) - f(x, y)]^2 + \lambda [f(x, y+1) - f(x, y)]^2 \}.$$

v. Suppose  $f$  minimizes  $E_{1,discrete}$ . Then for any  $(x_0, y_0) \in \Omega$ ,

$$\begin{aligned}
0 &= \frac{\partial E_{1,discrete}(f)}{\partial f(x_0, y_0)} \\
&= \frac{\partial}{\partial f(x_0, y_0)} \left( [f(x_0, y_0) - g(x_0, y_0)]^2 + \lambda \{ [f(x_0 + 1, y_0) - f(x_0, y_0)]^2 + [f(x_0, y_0) - f(x_0 - 1, y_0)]^2 \right. \\
&\quad \left. + [f(x_0, y_0 + 1) - f(x_0, y_0)]^2 + [f(x_0, y_0) - f(x_0, y_0 - 1)]^2 \right) \\
&= 2[f(x_0, y_0) - g(x_0, y_0)] + 2\lambda[4f(x_0, y_0) - f(x_0 + 1, y_0) - f(x_0 - 1, y_0) - f(x_0, y_0 + 1) \\
&\quad - f(x_0, y_0 - 1)],
\end{aligned}$$

i.e.  $(1+4\lambda)f(x_0, y_0) - \lambda f(x_0 + 1, y_0) - \lambda f(x_0 - 1, y_0) - \lambda f(x_0, y_0 + 1) - \lambda f(x_0, y_0 - 1) = g(x_0, y_0)$ .

vi. The gradient of  $E_{1,discrete}$  has been computed in the previous part. Hence the explicit gradient descent scheme for minimizing  $E_{1,discrete}$  with time step  $\tau > 0$  is given by:

$$\begin{aligned}
&\frac{f^{n+1}(x, y) - f^n(x, y)}{\tau} \\
&= -2(1 + 4\lambda)f^n(x, y) - 2\lambda[f^n(x + 1, y) + f^n(x - 1, y) + f^n(x, y + 1) + f^n(x, y - 1)] + 2g(x, y).
\end{aligned}$$

(b) i. Suppose  $f$  is a minimizer of  $E_2$ . Then for any  $v : \Omega \rightarrow \mathbb{R}$ ,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \Big|_{t=0} E_2(f + tv) \\
&= \int_{\Omega} \frac{\partial}{\partial t} \Big|_{t=0} [(f + tv - g)^2 + K \|\nabla(f + tv)\|^4] dx dy \\
&= \int_{\Omega} [2v(f - g) + K \frac{\partial}{\partial t} \Big|_{t=0} (\langle \nabla f + t \nabla v, \nabla f + t \nabla v \rangle)^2] dx dy \\
&= \int_{\Omega} [2v(f - g) + 2K \|\nabla f\|^2 \langle \nabla f, \nabla v \rangle] dx dy \\
&= 2 \int_{\Omega} v[f - g - \nabla \cdot (K \|\nabla f\|^2 \nabla f)] dx dy + 2 \int_{\partial\Omega} v K \|\nabla f\|^2 \langle \nabla f, \vec{n} \rangle ds.
\end{aligned}$$

Since  $v$  is arbitrarily chosen,  $f$  satisfies:

$$\begin{cases} -\nabla \cdot (K \|\nabla f\|^2 \nabla f)(x, y) + f(x, y) = g(x, y) & \text{if } (x, y) \in \Omega, \\ K(x, y) \|\nabla f(x, y)\|^2 \langle \nabla f(x, y), \vec{n}(x, y) \rangle = 0 & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

iii. From i., a descent direction is given by

$$v(x, y) = \begin{cases} -f(x, y) + g(x, y) + \nabla \cdot (K \|\nabla f\|^2 \nabla f)(x, y) & \text{if } (x, y) \in \Omega, \\ -K(x, y) \|\nabla f(x, y)\|^2 \langle \nabla f(x, y), \vec{n}(x, y) \rangle & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

Hence given  $f : \Omega \rightarrow \mathbb{R}$ , the gradient descent scheme for minimizing  $E_2$  is given by:

$$\frac{\partial f(x, y; t)}{\partial t} = \begin{cases} -f(x, y) + g(x, y) + \nabla \cdot (K \|\nabla f\|^2 \nabla f)(x, y) & \text{if } (x, y) \in \Omega, \\ -K(x, y) \|\nabla f(x, y)\|^2 \langle \nabla f(x, y), \vec{n}(x, y) \rangle & \text{if } (x, y) \in \partial\Omega. \end{cases}$$

Upon discretization in time and space, the explicit gradient descent scheme with time step  $\tau$  becomes:

$$\frac{f^{n+1}(x, y) - f^n(x, y)}{\tau} = \begin{cases} -f^n(x, y) + g(x, y) + \mathcal{D}_2(K \|\mathcal{D}_1(f^n)\|^2 \mathcal{D}_1(f^n))(x, y) & \text{if } (x, y) \in \Omega, \\ -K(x, y) \|\mathcal{D}_1(f)(x, y)\|^2 \langle \nabla f(x, y), \vec{n}(x, y) \rangle & \text{if } (x, y) \in \partial\Omega, \end{cases}$$

where the finite difference operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  approximate the differential operators  $\nabla$  and  $\nabla \cdot$ .