Chapter 6

Image Segmentation

In Chapter 6, we will discuss an important image processing problem, namely, the image segmentation problem. The main objective of image segmentation is to extract important objects from an image (usually done by extracting the boundaries of objects in an image). In this chapter, we will discuss two famous image segmentation models. They are: Active contour segmentation model and Chan-Vese segmentation model.

6.1 Basic idea of Image segmentation

Definition 6.1. Let f be an image. Image segmentation refers to the process to:

1. extract set of points representing the edges / boundaries of objects, or;
2. obtain a function $\varphi : \Omega \to \mathbb{R}$ such that the set: $\varphi^{-1}(\{0\}) = \{x \in \Omega : \varphi(x) = 0\}$ represents the edges of objects (implicit representation of edges), or;
3. obtain a "black and white" image such that the white color regions represent the objects.

Edge detector

Definition 6.2. Edge detector $V : \Omega \to \mathbb{R}$ is a function defined on the image domain Ω such that $V(\varphi)$ is small at pixels which are on the edges / boundaries.

Examples of edge detectors

Example 6.3. $V(\vec{x}) = -|\nabla I(\vec{x})|$, $\vec{x} \in \Omega =$ image domain.

It is the most common used edge detector. In particular, $V(\vec{x}) = 0$ on homogeneous region (i.e. region with uniform / constant image intensity).

On the edge, there is a big jump in image intensity. Thus, $V(\vec{x}) = -|\nabla I(\vec{x})|$ is very small (negative).

In the discrete case, gradient can be computed by spatial linear filtering with filters:

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ in } x \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ in } y$$

Example 6.4. $V(\vec{x}) = \dfrac{1}{1+|\nabla I(\vec{x})|}$, $\vec{x} \in \Omega =$ image domain.

What are the common image segmentation models?

1. Active contour model: Parameterized curve evolution

   Goal: Find a parameterized curve $\gamma : [0, 2\pi] \to \Omega$ such that it represents the boundary.

2. Level set model: Implicit representation of contour curves capturing the boundaries of the objects.

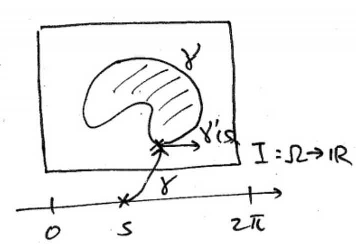
   Goal: Find a function $\varphi : \Omega \to \mathbb{R}$ (Ω is the image domain) such that the zero level set $\varphi^{-1}(\{0\}) = \{x \in \Omega : \varphi(x) = 0\}$ represents the boundaries of the objects.

6.2 Active contour model

(Proposed by Kass, Witkin, Terzopoulous)

Let $I : \Omega \to \mathbb{R}$ be an image defined on an image domain Ω.

Our goal: find a parameterized curve $\gamma : [0, 2\pi] \to \Omega \subset \mathbb{R}^2$, which is the boundary of an object. Assume that the curve is closed.



Given that V is the edge detector such that V is very small on the boundary of the object in an image.

Technique: Find smooth curve such that $V(\gamma(s))$ is small for every $0 \le s \le 2\pi$.

Consider the snake energy:

$$E_{snake}(\gamma) = \int_0^{2\pi} \frac{1}{2}|\gamma'(s)|^2 \, ds \; + \beta \int_0^{2\pi} V(\gamma(s)) \, ds$$

Remark.

1. *First term minimizes the first derivative, hence, enhances the smoothness of* $\gamma(s)$.

2. *Second term minimizes* $V(\gamma(s))$*, attracting* γ *to the boundary of the object.*

Mathematical problem:

Segmentation is equivalent to finding an optimal curve $= \underset{\gamma:[0,2\pi]\to\Omega}{\operatorname{argmin}} \; E(\gamma)$

We use gradient descent to minimize $E_{snake}(\gamma)$.

Given a contour $\gamma^n(t)$ at the n-th iteration. We proceed to perturb $\gamma^{n+1}(t)$ by $\gamma^{n+1} := \gamma^n + \varepsilon\varphi$ to minimize E_{snake}. We need to find φ s.t. $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^{n+1}) < 0$.

$$
\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^{n+1}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}|(\gamma^n)'(s) + \varepsilon\varphi'(s)|^2 \, ds \ + \beta \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} V(\gamma^n(s) + \varepsilon\varphi(s)) \, ds \\
&= \int_0^{2\pi} (\gamma^n)'(s) \cdot \varphi'(s) \, ds \ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s)) \cdot \varphi(s) \, ds \\
&= -\int_0^{2\pi} (\gamma^n)''(s) \cdot \varphi(s) \, ds \ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s)) \cdot \varphi(s) \, ds \\
&= \int_0^{2\pi} (-(\gamma^n)''(s) + \beta\nabla V(\gamma^n(s))) \cdot \varphi(s) \, ds
\end{aligned}
$$

In order that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake}(\gamma^{n+1}) < 0$ (decreasing), we must have:

$$\varphi(s) = (\gamma^n)''(s) - \beta\nabla V(\gamma^n(s))$$

Thus, we must modify γ^n by:

$$\gamma^{n+1} = \gamma^n + \varepsilon((\gamma^n)''(s) - \beta\nabla V(\gamma^n(s))) \qquad \text{for small } \varepsilon > 0$$

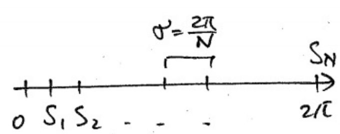
or

$$\frac{\gamma^{n+1} - \gamma^n}{\varepsilon} = \underbrace{(\gamma^n)''(s) - \beta\nabla V(\gamma^n(s))}_{-\nabla E(\gamma^n(s)) \ (definition)}$$

In the continuous setting, we aim to obtain a time-dependent contour: $\gamma_t(s) := \gamma(s;t)$ such that: $\frac{d}{dt}\gamma_t(s) = -\nabla E(\gamma_t(s))$.

<u>**Discretization of the snake model**</u>:

Let N = number of discrete points in $[0, 2\pi]$,
$\sigma = \frac{2\pi}{N}$ = step length and $s_i = i\sigma \quad (i = 1, 2, \cdots, N)$
$u_i^k = \gamma(s_i; t^k) = \gamma(i\sigma; k\tau)$ = i-th node of the contour.



<u>**Define**</u>: $u^k = (u_1^k, u_2^k, \cdots, u_N^k)^T \in M_{N\times 2}(\mathbb{R})$ = discrete closed curve / contour, where $u_i^k \in \mathbb{R}^2$ for all i.

The discrete derivative can be approximated by finite difference scheme:

$$\gamma_k'(i\sigma) = \frac{u_{i+1}^k - u_i^k}{\sigma} \qquad i = 1, 2, \cdots, N$$

Here, we assume $\gamma_{N+1}^k = \gamma_1^k$ and $\gamma_N^k = \gamma_0^k$ (since the contour is closed).

Thus, the discrete snake energy can be written as:

$$E_{snake}(u) = \sum_{i=1}^{N} \frac{1}{2} \left| \frac{u_{i+1} - u_i}{\sigma} \right|^2 \sigma + \beta \sum_{i=1}^{N} V(u_i)\sigma$$

where $u = (u_1, u_2, \cdots, u_N)^T$ is a discrete closed curve ($u_i \in \mathbb{R}^2$ for all i). We can throw away σ to obtain:

$$E_{snake}(u) = \sum_{i=1}^{N} \frac{1}{2} \left| \frac{u_{i+1} - u_i}{\sigma} \right|^2 + \beta \sum_{i=1}^{N} V(u_i)$$

To minimize E_{snake}, we compute ∇E and find u such that $\nabla E(u) = 0$. now,

$$\frac{\partial E}{\partial u_i} = -\left(\frac{u_{i+1} - u_i}{\sigma^2}\right) + \left(\frac{u_i - u_{i-1}}{\sigma^2}\right) + \beta \nabla V(u_i)$$

(Recall that: $u_i = (u_{i_1}, u_{i_2})^T \in \mathbb{R}^2$. We define: $\dfrac{\partial E}{\partial u_i} = \left(\dfrac{\partial E}{\partial u_{i_1}}, \dfrac{\partial E}{\partial u_{i_2}}\right)^T$.

Thus, $\dfrac{\partial V}{\partial u_i} = \left(\dfrac{\partial V}{\partial u_{i_1}}, \dfrac{\partial V}{\partial u_{i_2}}\right)^T = \nabla V(u_i))$

Define: $D = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}$ (Recall: $u_{N+1} = u_1$ and $u_0 = u_N$) and

define $F(u) = (F_1(u), F_2(u), \cdots, F_N(u))^T \in M_{N\times 2}(\mathbb{R})$

where $F_i(u) = -\nabla V(u_i), \quad i = 1, 2, \cdots, N$.

Then: $\dfrac{\partial E}{\partial u_i} = -\dfrac{(Du)_i}{\sigma^2} - \beta(F(u))_i$

Using the gradient descent method, we can minimize E_{snake} by:

1. Explicit Euler scheme:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \frac{(Du^k)_i}{\sigma^2} + \beta(F(u^k))_i \qquad \tau = \text{ time step}$$

2. Semi-implicit scheme:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \frac{(Du^{k+1})_i}{\sigma^2} + \beta(F(u^k))_i \qquad \tau = \text{ time step}$$

For the semi-implicit scheme, the equation can be re-written as:

$$\left(I - \frac{\tau}{\sigma^2} D\right) u^{k+1} = u^k + \beta \tau F(u^k)$$

Remark.

1. *τ and σ has to be carefully chosen.*

2. *$F(u^k)$ is called the force driving the discrete contour to the boundary.*

3. *$(F(u))_i = -\nabla V(u_i)$. Usually, the edge detector is smoothed out by $\tilde{V} = G * V$ where G is a Gaussian function.*

4. *Active contour model cannot allow topological change of the contour. (Allow one closed curve to capture one boundary)*

5. *Active contour is sensitive to noise, since it depends on the edge detector.*

Example 6.5. Let $I : \Omega \to \mathbb{R}$ be an image and $V : \Omega \to \mathbb{R}$ is the edge detector defined by:

$$V(\vec{p}) = V((p_1, p_2)) = \begin{cases} p_1^2 + p_2^2 & \text{if } p_1^2 + p_2^2 \geq 1 \\ 1 & \text{if } p_1^2 + p_2^2 < 1 \end{cases}$$

Assuming that the initial contour encloses the unit circle. Explain (intuitively) why the active contour model converges to a circle $C : \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}$.

Solution. We consider the explicit Euler model first.

$$\frac{u^{n+1} - u^n}{\tau} = \frac{Du^n}{\sigma^2} + \alpha F(u^n)$$

where $(F(u))_i = -\nabla V(u_i) = -\nabla V((u_{i1}, u_{i2})) = -2(u_{i1}, u_{i2})$.

Hence, if $u^n = \begin{pmatrix} u_{11}^n & u_{12}^n \\ u_{21}^n & u_{22}^n \\ \vdots & \vdots \\ u_{N1}^n & u_{N2}^n \end{pmatrix}$, then $F(u^n) = -2 \begin{pmatrix} u_{11}^n & u_{12}^n \\ u_{21}^n & u_{22}^n \\ \vdots & \vdots \\ u_{N1}^n & u_{N2}^n \end{pmatrix}$ (contour force). Our model becomes

$$u^{n+1} = u^n + \tau \frac{Du^n}{\sigma^2} - 2\alpha\tau \begin{pmatrix} u_{11}^n & u_{12}^n \\ u_{21}^n & u_{22}^n \\ \vdots & \vdots \\ u_{N1}^n & u_{N2}^n \end{pmatrix}$$

As $(F(u^n))_i = -\nabla V(u_{i1}^n, u_{i2}^n) = -2(u_{i1}^n, u_{i2}^n)$, the force attracts the contour to the circle. Also, $F = \vec{0}$ in the interior region since $\nabla V = 0$ (V = constant in the interior region C).

Now, $u^{n+1} = (1 - 2\alpha\tau) \begin{pmatrix} u_{11}^n & u_{12}^n \\ u_{21}^n & u_{22}^n \\ \vdots & \vdots \\ u_{N1}^n & u_{N2}^n \end{pmatrix} + \tau \frac{Du^n}{\sigma^2}$.

For the semi-implicit scheme, the model is given by:

$$\frac{u^{n+1} - u^n}{\tau} = \frac{Du^{n+1}}{\sigma^2} + \alpha F(u^n)$$

Hence, we have $\left(I - \frac{\tau}{\sigma^2}D\right) u^{n+1} = (1-2\alpha\tau)u^n$.

If τ is comparatively small, $\left(I - \frac{\tau}{\sigma^2}D\right) \approx I$.

The right hand side draws the contour to the unit circle.

Example 6.6. Consider the following discrete curve evolution model:

$$\frac{u^{n+1}-u^n}{\tau} = Au^n + \alpha F(u^n)$$

where $(F(u))_i = -\nabla V(u_i)$ and $V(\vec{x}) = V((x,y)) = x^2 + y^2$,

$A = \begin{pmatrix} \sigma(1) & & & \\ & \sigma(2) & & \\ & & \ddots & \\ & & & \sigma(N) \end{pmatrix}$ and $\sigma(j) < 0$ for all j.

Prove that $\{u^n\}_{n=1}^{\infty}$ converges to a point curve $u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$.

Solution.

$$\begin{aligned} u^{n+1} &= \left(I + \tau \begin{pmatrix} \sigma(1) & & & \\ & \sigma(2) & & \\ & & \ddots & \\ & & & \sigma(N) \end{pmatrix} - 2\alpha\tau I \right) u^n \\ &= \begin{pmatrix} 1+\tau\sigma(1)-2\alpha\tau & & & \\ & 1+\tau\sigma(2)-2\alpha\tau & & \\ & & \ddots & \\ & & & 1+\tau\sigma(N)-2\alpha\tau \end{pmatrix} u^n \end{aligned}$$

Assume τ is small enough such that $1+\tau\sigma(j)-2\alpha\tau > 0$ for all j.
Then, $0 < 1 + \underbrace{\tau\sigma(j)}_{-ve} - 2\alpha\tau < 1$ for all j. Easy to check:

$$u^n = \begin{pmatrix} \underbrace{(1+\tau\sigma(1)-2\alpha\tau)^n}_{\to 0} & & & \\ & \underbrace{(1+\tau\sigma(2)-2\alpha\tau)^n}_{\to 0} & & \\ & & \ddots & \\ & & & \underbrace{(1+\tau\sigma(N)-2\alpha\tau)^n}_{\to 0} \end{pmatrix} u^0$$

Then, $u^n \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ when $n \to \infty$.

For semi-implicit scheme, we have:

$$\frac{u^{n+1}-u^n}{\tau} = Au^{n+1} + \alpha F(u^n)$$

Then,

$$\left(I - \tau \begin{pmatrix} \sigma(1) & & & \\ & \sigma(2) & & \\ & & \ddots & \\ & & & \sigma(N) \end{pmatrix}\right) u^{n+1} = (1-2\alpha\tau)u^n$$

$$\Rightarrow u^{n+1} = (1-2\alpha\tau) \begin{pmatrix} \frac{1}{1-\tau\sigma(1)} & & & \\ & \frac{1}{1-\tau\sigma(2)} & & \\ & & \ddots & \\ & & & \frac{1}{1-\tau\sigma(N)} \end{pmatrix} u^n$$

$$\Rightarrow u^n = \begin{pmatrix} \left(\frac{1-2\alpha\tau}{1-\tau\sigma(1)}\right)^n & & & \\ & \left(\frac{1-2\alpha\tau}{1-\tau\sigma(2)}\right)^n & & \\ & & \ddots & \\ & & & \left(\frac{1-2\alpha\tau}{1-\tau\sigma(N)}\right)^n \end{pmatrix} u^0$$

$$\therefore u^n \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \text{ if } 1-2\alpha\tau > 0$$

since $1-\tau\sigma(j) > 1$ for all j.

More flexibility in choosing the time step τ.

For explicit Euler scheme, if τ is too big such that $1+\tau\sigma(j)-2\alpha\tau < -1$, then $|1+\tau\sigma(j)-2\alpha\tau| > 1$ and hence $(1+\tau\sigma(j)-2\alpha\tau)^n$ does not converge when $n \to \infty$. Hence, u^n cannot converge if τ is not carefully chosen.

Remark.

- *Explicit Euler scheme is relatively simpler, as it does not involve finding the matrix inverse. Just involve matrix multiplication in each iteration.*
- *But the convergence of explicit Euler scheme relies on τ.*
- *Semi-implicit scheme involves the computation of matrix inverse:*

$$u^{n+1} = \left(1 - \frac{\tau D}{\sigma^2}\right)^{-1} (u^n + \tau\alpha F(u^n))$$

- *But semi-implicit scheme has more flexibility when choosing τ. It (in most situations) allows larger τ to ensure convergence.*

6.3 Implicit model for image segmentation (Optional)

6.3.1 Implicit representation of curves

Let $I : \Omega \to \mathbb{R}$ be an image. Let $\varphi : \Omega \to \mathbb{R}$ be a real-valued function defined on Ω. Suppose Ω_0 is the object in Ω, we may assume that $\varphi(z) > 0$ inside Ω_0 and $\varphi(z) < 0$ outside Ω_0 (i.e. in $\Omega \setminus \Omega_0$), Then, the zero-level set $\varphi^{-1}(\{0\}) = \{\vec{p} \in \Omega : \varphi(\vec{p}) = 0\}$ is a curve in Ω, which reoresents the boundary of the object. This is called the **level set representation** of the curve and φ is called the level set function.

Computation of the level set function

Let Ω_0 be the object. Want: $\varphi : \Omega \to \mathbb{R}$ representing Ω_0. Suppose we know the inside and outside regions of Ω_0. Let $d(\vec{x}, \partial\Omega_0) = \inf_{\vec{y} \in \partial\Omega_0} ||\vec{x} - \vec{y}||$ = minimal distance from $\vec{x}$ to the boundary of Ω_0, $\partial\Omega_0$. We can define the level set function as:

$$\varphi(\vec{x}) = \begin{cases} d(\vec{x}, \partial\Omega_0) \text{ if } \vec{x} \in \text{ inside} \\ -d(\vec{x}, \partial\Omega_0) \text{ if } \vec{x} \in \text{ outside} \end{cases}$$

Obviously, $\varphi^{-1}(\{0\}) = \partial\Omega_0$.

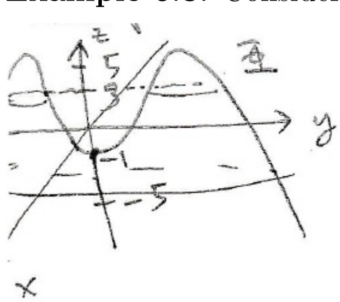
Remark.
- *Working with the level set function for curve evolution increases the dimension of the problem (from 1D to 2D).*
- *If $\varphi : \mathbb{R}^3 \to \mathbb{R}$, then $\varphi^{-1}(\{0\})$ is a surface embedded in $\mathbb{R}^3$ (implicit function theorem).*

Example 6.7. Consider: $\varphi(\vec{x}) = 2 - (x^2 + y^2)$. Then, $\varphi^{-1}(\{0\})$ = circle centred at the origin with radius = $\sqrt{2}$.

To expand the circle, we consider: $\varphi(\vec{x}) = (2 + \varepsilon) - (x^2 + y^2)$. Then, $\varphi_\varepsilon^{-1}(\{0\})$ = circle centred at the origin with radius = $\sqrt{2 + \varepsilon}$ ($> \sqrt{2}$).

Remark. *By modifying φ, we can evolve a curve.*

Example 6.8. Consider a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ which looks like:



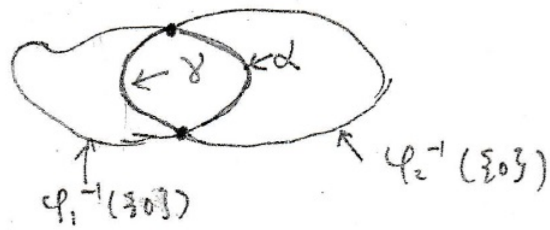
Let $\varphi_K(x, y) = K - \Phi(x, y)$. Then,

$$\begin{aligned} \varphi_{-5}^{-1}(\{0\}) &= \text{ one simple closed curve} \\ \varphi_{-1}^{-1}(\{0\}) &= \text{ two simple closed curves touching each other} \\ \varphi_{3}^{-1}(\{0\}) &= \text{ two separated simple closed curve} \end{aligned}$$

Remark. *Level set method allows topological change. Therefore, level set based segmentation algorithm can evolve one closed curve to several closed curves capturing the boundaries of multiple objects.*

Question 1: How to represent open curve?

Answer: We usually need two level set functions. The curve $\gamma = \varphi_2^{-1}(\{0\}) \cap \{\varphi_1 > 0\}$ ($\{\varphi_1 > 0\} = \{\vec{x} \in \Omega : \varphi_1(\vec{x}) > 0\}$). The curve $\alpha = \varphi_1^{-1}(\{0\}) \cap \{\varphi_2 > 0\}$



Question 2: How to represent a curve in $\mathbb{R}^3$?

Answer: We need two level set functions. Let $\varphi_1 : \mathbb{R}^3 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^3 \to \mathbb{R}$. Then, $\varphi_1^{-1}(\{0\}), \varphi_2^{-1}(\{0\})$ are two surfaces embedded in $\mathbb{R}^3$. Then, $\varphi_1^{-1}(\{0\}) \cap \varphi_2^{-1}(\{0\})$ is a curve in $\mathbb{R}^3$.

Level set method for image segmentation

Goal: Find a level set function φ that minimizes:

$$E(\varphi) = \underbrace{F_1(\varphi)}_{\text{smoothness}} + \underbrace{F_2(\varphi)}_{\text{data ficlelity}}$$

Assumption:

1. $F_1(\varphi)$ = smoothness term. Smoothness is achieved by minimizing the length of $\varphi^{-1}(\{0\})$.
2. $F_2(\varphi)$ = want $\{\varphi > 0\}$ matches with the object in the image. We assume the image is piecewise constant. Therefore, each object has uniform (constant) image intensity.

Geometric quantities obtained from level set representation

Outward unit normal: $\dfrac{\nabla\phi(\vec{x})}{|\nabla\phi(\vec{x})|}$, $\vec{x} \in \Gamma = \phi^{-1}(\{0\})$

Curvature of contour: $\kappa(\vec{x}) = div\left(\dfrac{\nabla\phi(\vec{x})}{|\nabla\phi(\vec{x})|}\right)$; $\vec{x} \in \Gamma$

Curve length element ds: $ds = \delta(\phi(\vec{x}))|\nabla\phi(\vec{x})|d\vec{x}$, where δ = delta function.

6.3.2 Chan-Vese Segmentation model / Active contour without edge

Minimize: $E(\varphi) = F_1(\varphi) + F_2(\varphi)$

For $F_1(\varphi)$, we choose it as:

$$\begin{aligned} Length(\Gamma) &= \int_\Gamma d\sigma \\ &= \int \delta(\varphi(\vec{x}))|\nabla\phi(\vec{x})|d\vec{x} \\ &= \int |\nabla H(\varphi(\vec{x}))|d\vec{x} \end{aligned}$$

where $H(x) = \text{Heaviside function} = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ and $\nabla H(\varphi(\vec{x}))$ means the gradient of $H \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$ at $\vec{x}$. (Coarea formula)

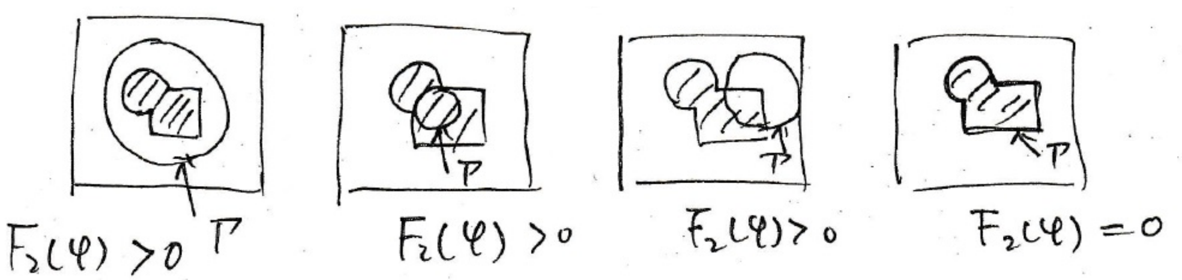
Usually, a smooth approxiamtion of $H(x)$ is used (e.g. $H_k(x) = \frac{1}{1+e^{-2kx}}$)

For $F_2(\varphi)$, we use:

$$\int_{\text{inside } \Gamma} |I(x,y) - c_1|^2 \, dx \, dy + \int_{\text{outside } \Gamma} |I(x,y) - c_2|^2 \, dx \, dy$$

where c_1 = average I inside Γ, c_2 = average I outside Γ.

<u>Observe</u>:



Therefore, the overall Chan-Vese Segmentation model: Find level set function φ and intensities c_1, c_2 such that they minimize:

$$E(\varphi, c_1, c_2) = \int_\Omega |\nabla H(\varphi(\vec{x}))|d\vec{x} + \lambda \int_{\{\varphi>0\}} |I(x,y) - c_1|^2 \, dx \, dy + \lambda \int_{\{\varphi<0\}} |I(x,y) - c_2|^2 \, dx \, dy$$

Again, $\nabla H(\varphi(\vec{x}))$ means the gradient of $H \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$ at $\vec{x}$.

If φ is fixed, the contour $\Gamma = \varphi^{-1}\{0\}$ is fixed. Then,

$$0 = \frac{\partial}{\partial c_1} E(\Gamma, c_1, c_2) = -2 \int_{\{\varphi>0\}} (I(x,y) - c_1) \, dx \, dy$$

Therefore, $c_1 = \dfrac{\int_{\{\varphi>0\}} I(x,y) \, dx \, dy}{\int_{\{\varphi>0\}} \, dx \, dy} = \dfrac{\int_\Omega H(\varphi) I(x,y) \, dx \, dy}{\int_\Omega H(\varphi) \, dx \, dy}$.

Similarly, $c_2 = \dfrac{\int_{\{\varphi<0\}} I(x,y) \, dx \, dy}{\int_{\{\varphi<0\}} \, dx \, dy} = \dfrac{\int_\Omega (1 - H(\varphi)) I(x,y) \, dx \, dy}{\int_\Omega (1 - H(\varphi)) \, dx \, dy}$

Now, fixing c_1, c_2, we proceed to look for a time-dependent level set function $\varphi = \varphi(t, x, y)$ that minimizes $E(\varphi)$.

Using gradient descent method, we get

$$\frac{\partial \varphi}{\partial t} = \underbrace{\delta(\varphi)}_{\text{smooth approx.}} \left[\underbrace{\nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|} \right)}_{\text{curvature}} - \lambda (I - c_1)^2 + \lambda (I - c_2)^2 \right],$$

where $\varphi(0, x, y) = \varphi_0(x, y) =$ initial prescribed level set function.

We leave it as an exercise to check the formula.