Tutorial 5

$*$ More about limits

① Consider the following example in Week 5 lecture note:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) \underset{\uparrow}{=} 0$$

(by Squeeze Theorem)

(a) What does $\lim_{r\to 0} r(\cos^3\theta+\sin^3\theta)=0$ mean?

1st choice: For each (fixed) $\theta\in\mathbb{R}$, $\lim_{r\to 0} \underbrace{r(\cos^3\theta+\sin^3\theta)}=0$.

considered as $f:\mathbb{R}\to\mathbb{R}$, and use ε-δ definition in Week 4 note as the meaning

2nd choice: $\lim_{(r,\theta)\to(0,0)} \underbrace{r(\cos^3\theta+\sin^3\theta)} = 0$

considered as $g:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ and use ε-δ definition in Week 4 note as the meaning

3rd choice: other meaning

(b) How to get $\lim_{r\to 0} r(\cos^3\theta+\sin^3\theta)=0$?

(What kind of Squeeze Theorem is being used here?)

"Ans": 1(a) Probably the 3rd choice.

Let $x,y\in\mathbb{R}$. There exist $r\in[0,\infty)$ and $\theta\in\mathbb{R}$ such that $x=r\cos\theta$ and $y=r\sin\theta$.

Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function.

$\lim_{r\to 0} f(r\cos\theta, r\sin\theta)=L$ probably means

$\forall\varepsilon>0$, $\exists\delta>0$ s.t. if $r\in(-\delta,\delta)\cap[0,\infty)\setminus\{0\}$ and $\theta\in\mathbb{R}$

then $|f(r\cos\theta, r\sin\theta)-L|<\varepsilon$.

Remark: $\displaystyle \lim_{(x,y)\to(0,0)} f(x,y) = L \iff \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$ (*)

(Usual def.)
$\forall \varepsilon>0, \exists \delta>0$ s.t.
if $(x,y) \in B_\delta((0,0)) \cap \mathbb{R}^2 \setminus \{(0,0)\}$,
then $|f(x,y) - L| < \varepsilon$.

("new" def.)
$\forall \varepsilon>0, \exists \delta>0$ s.t.
if $r \in (-\delta, \delta) \cap [0,\infty) \setminus \{0\}$ and $\theta \in \mathbb{R}$,
then $|f(r\cos\theta, r\sin\theta) - L| < \varepsilon$

"Ans": 1(b)

Given the "new" definition of $\displaystyle \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0$,
Can I still use Squeeze Theorem to argue $\displaystyle \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$?

We may need a "modified version of Squeeze Theorem":

If $f: \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}$ and $g, h: (0,\infty) \to \mathbb{R}$
are functions such that
$(\forall r \in (0,\infty), \forall \theta \in \mathbb{R}, \ g(r) \le f(r\cos\theta, r\sin\theta) \le h(r))$
and $\displaystyle \lim_{r\to 0} g(r) = L = \lim_{r\to 0} h(r)$,
then $\displaystyle \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$

usual definition

"new" definition

Since $-2|r| \le r(\cos^3\theta + \sin^3\theta) \le 2|r|$
and $\displaystyle \lim_{r\to 0}(-2|r|) = 0 = \lim_{r\to 0} 2|r|$,
by the "modified Squeeze Theorem", $\displaystyle \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$.

(Thus, by (*), $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$)

② Let $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined as

$$f(x,y) = \frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}.$$

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Ans: Let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. There exist $r \in [0,\infty)$ and $\theta \in \mathbb{R}$ such that $x = r\cos\theta$ and $y = r\sin\theta$.

$$f(x,y) = \frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2} = \frac{3r^2 - (r\cos\theta)^2(r\sin\theta)^2}{r^2}$$

$$= 3 - r^2\cos^2\theta\sin^2\theta$$

$$= f(r\cos\theta, r\sin\theta)$$

$$|r^2\cos^2\theta\sin^2\theta| = |r^2|\cdot|\cos^2\theta|\cdot|\sin^2\theta| \leq r^2$$

$$-r^2 \leq r^2\cos^2\theta\sin^2\theta \leq r^2$$

$$r^2 \geq -r^2\cos^2\theta\sin^2\theta \geq -r^2$$

$$3+r^2 \geq 3 - r^2\cos^2\theta\sin^2\theta \geq 3-r^2$$

$$\lim_{r\to 0}(3+r^2) = 3 = \lim_{r\to 0}(3-r^2)$$

By Squeeze Theorem, $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 3$
(the modified one in 1(b))

By (*), $\lim_{(x,y)\to(0,0)} f(x,y) = 3$.

③ (a) Let $f(x) = \begin{cases} (x-1)^2 & \text{if } x > 1 \\ (x-1)^3 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \end{cases}$. Does $\lim\limits_{x \to 1} f(x)$ exist?

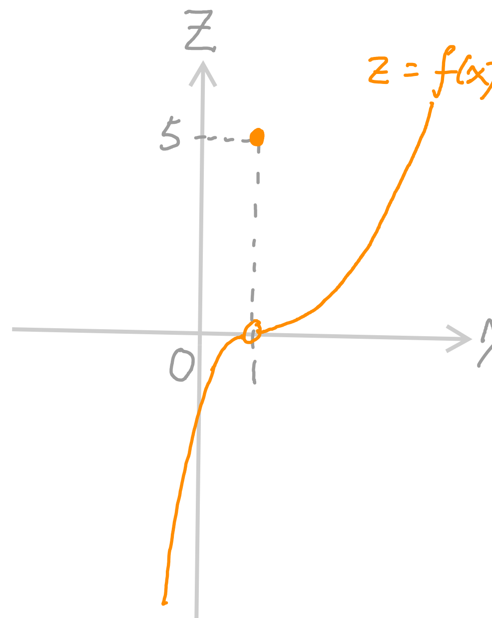
(b) Let $f(x,y) = \begin{cases} (x-1)^2 & \text{if } x > 1 \\ (x-1)^3 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \end{cases}$. Does $\lim\limits_{(x,y) \to (1,0)} f(x,y)$ exist?

Ans: (a) Yes.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1)^2 = 0$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x-1)^3 = 0$$

$\lim\limits_{x \to 1} f(x)$ exists (and equal 0)



(b) No.

$$\lim_{\substack{(x,y) \to (1,0) \\ \text{along } x=1}} f(x,y) = \lim_{\substack{(x,y) \to (1,0) \\ \text{along } x=1}} 5 = 5$$

$$\lim_{\substack{(x,y) \to (1,0) \\ \text{along } y=0}} f(x,y) = 0 \neq 5$$

$\therefore \lim\limits_{(x,y) \to (1,0)} f(x,y)$ does not exist.