Solution to Homework 2

Sec. 2.5

2. (d) We look for $[I]_{\beta'}^{\beta}$. Let γ be the standard basis for $\mathbb{R}^2$. First we can find the change of coordinate matrices that changes β and β' to γ.

$$[I]_{\beta}^{\gamma} = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \text{ and } [I]_{\beta'}^{\gamma} = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$$

Then computing $[I]_{\beta'}^{\beta}$ is easy. In particular, we change basis from β' to γ, then we change basis from γ to β.

$$[I]_{\beta'}^{\beta} = [I]_{\gamma}^{\beta}[I]_{\beta'}^{\gamma} = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$$

3. (f) Again we first change from basis β' to standard basis γ for $P_2(\mathbb{R})$. Then we change basis from γ to β. First we can find the change of coordinate matrices that changes β and β' to γ.

$$[I]_{\beta}^{\gamma} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix} \text{ and } [I]_{\beta'}^{\gamma} = \begin{pmatrix} 0 & 1 & 3 \\ 9 & 21 & 5 \\ -9 & -2 & 2 \end{pmatrix}$$

Then computing $[I]_{\beta'}^{\beta}$ is easy. In particular, we change basis from β' to γ, then we change basis from γ to β.

$$\begin{aligned} [I]_{\beta'}^{\beta} = [I]_{\gamma}^{\beta}[I]_{\beta'}^{\gamma} &= \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 3 \\ 9 & 21 & 5 \\ -9 & -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix} \end{aligned}$$

4. We want to find $[T]_{\beta'}$. Through basis β, we have

$$[T]_{\beta'} = [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta}.$$

Now $[T]_\beta$ is simply $\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$. Since β is just the standard basis for $\mathbb{R}^2$, we have

$$[I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

By the fact, we get

$$[I]_{\beta}^{\beta'} = \left([I^{-1}]_{\beta'}^{\beta}\right)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Finally we can compute

$$[T]_{\beta'} = [I]_{\beta}^{\beta'}[T]_\beta[I]_{\beta'}^{\beta} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & 9 \end{pmatrix}$$

6. (d) Let γ be the standard basis for $\mathbb{F}^3$. One can follow the approach that

$$[L_A]_\beta = [I]_{\gamma}^{\beta}[L_A]_\gamma[I]_{\beta}^{\gamma}.$$

As $[L_A]_\gamma$ to be just A, we see that Q is $[I]_{\beta}^{\gamma}$.

$$Q = [I]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

We may compute $[L_A]_\gamma$ as described. However we observe that

$$A\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},$$

$$A\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} = 12\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = 18\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

By definition, $[L_A]_\gamma$ is just $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$.

11. (a) Note that $Q = [I]_{\alpha}^{\beta}$ and $R = [I]_{\beta}^{\gamma}$. Then $RQ = [I]_{\beta}^{\gamma}[I]_{\alpha}^{\beta} = [I]_{\alpha}^{\gamma}$ changes α-coordinates into β-coordinates.

(b) Since $Q = [I]_{\alpha}^{\beta}$, we have $Q^{-1} = \left([I]_{\alpha}^{\beta}\right)^{-1} = [I^{-1}]_{\beta}^{\alpha} = [I]_{\beta}^{\alpha}$.

13. To prove $\beta' = \{x_1', x_2', \ldots, x_n'\}$ is a basis, it suffices to show that β' are linearly independent as it has the same size as the basis β. Now suppose

$$\sum_{j=1}^{n} a_j x_j' = 0,$$

we want to show $a_j = 0$ for $j = 1, 2, \ldots, n$. By definition,

$$\sum_{j=1}^{n} a_j x_j' = \sum_{j=1}^{n} a_j \sum_{i=1}^{n} Q_{ij} x_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_j Q_{ij} \right) x_i = 0.$$

By the linear independence of β, we have $\sum_{j=1}^{n} a_j Q_{ij} = 0$ for each $i = 1, 2, \ldots, n$. But that just mean that

$$\begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As Q is invertible, $a_1 = a_2 = \cdots = a_n = 0$ is the only solution. This proves β' are linearly independent, hence a basis.

To see Q changes β'-coordinates into β-coordinate, we observe that

$$Qe_j = \begin{pmatrix} Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{nj} \end{pmatrix},$$

where e_j is a vector with a 1 at the jth entry and 0's elsewhere. As $x_j' = \sum_{i=1}^{n} Q_{ij} x_i$, Q is the change of coordinate matrix.

Sec. 5.1

2. (d) Check that

$$\begin{cases} T(x - x^2) &= 4 + 4x - 4x^2 &= -4(-1 - x + x^2), \\ T(-1 + x^2) &= 2 - 2x^2 &= -2(-1 + x^2), \\ T(-1 - x + x^2) &= 3x - 3x^2 &= 3(x - x^2). \end{cases}$$

Hence, we have $[T]_\beta = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{pmatrix}$. Also, we see that only $-1 + x^2$ is an eigenvector corresponding to eigenvalue -2. It is easy to check that it is a basis, but it is consisting of only one eigenvector.

(f) Again, we check for each vector under transformation.

$$T\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}-3 & 0\\-3 & 0\end{pmatrix} = -3\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$$
$$T\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix} = \begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix}$$
$$T\begin{pmatrix}1 & 0\\2 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\2 & 0\end{pmatrix}$$
$$T\begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}$$

Hence, we have $[T]_\beta = \begin{pmatrix}-3 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$.

From the above, we see that each vector is an eigenvector. Eigenvalues are -3, 1, 1 and 1 respectively. Obviously, β is a basis, hence it is a basis consisting of eigenvectors of T.

6. Note that λ is an eigenvalue with respect to some eigenvector v if and only if

$$Tv = \lambda v.$$

Since β is a basis, this is equivalent to $[Tv]_\beta = [\lambda v]_\beta$, that is

$$[T]_\beta[v]_\beta = \lambda[v]_\beta.$$

Hence, it is true if and only if λ is the eigenvalue of $[T]_\beta$ with respect to $[v]_\beta$.

8. (a) Note that zero is an eigenvalue of T if and only if there exists some nonzero vector v such that $Tv = 0v = 0$. This is equivalent to say that there exists some nonzero vector

$$v \in N(T - \lambda I) = N(T),$$

that is $N(T) \neq \{\mathbf{0}\}$. So T is not invertible. In other words, T is invertible if and only if zero is not an eigenvalue of T.

(b) Again λ is an eigenvalue if and only if $T(v) = \lambda v$ for some nonzero vector v. As T is invertible, from the above, we see $\lambda \neq 0$. So this means

$$\lambda v = \lambda^{-1}T^{-1}(T(v)) = T^{-1}v,$$

which means that λ^{-1} is an eigenvalue of T^{-1}.

(c) i. First, we show that M is invertible if and only if $\lambda = 0$ is not an eigenvalue. This is true because λ is an eigenvalue if and only if there exists some nonzero vector v such that

$$Mv = \lambda v.$$

But λ is just zero, this means there is some nontrivial solution to the system

$$Mv = 0,$$

that is M is not invertible. In order words, M is invertible if and only if zero is not an eigenvalue of M.

ii. Second, we prove that λ^{-1} is an eigenvalue of M^{-1}. As M is invertible, we have $\lambda \neq 0$ by the above argument. Since there is some nonzero vector v such that $Mv = \lambda v$, hence we can multiply both sides by M^{-1} and λ^{-1}.

$$\lambda^{-1}v = \lambda^{-1}M^{-1}Mv = M^{-1}v$$

This is equivalent to say that λ^{-1} is an eigenvalue of M^{-1}.

16. (a) Note that from Exercise 13 in Sec. 2.3, we have $\text{tr}(AB) = \text{tr}(BA)$. Hence, if A is similar to B, there exists some invertible matric P such that $A = P^{-1}BP$. So

$$\text{tr}(A) = \text{tr}(P^{-1}BP) = \text{tr}(BPP^{-1}) = \text{tr}(B).$$

(b) As every linear operator on a finite-dimensional vector space could be represented as a matrix, hence it would be natural to define the trace of a linear operator as the trace of its matrix representation.

From the above part, we also know that this trace is independent of the choice of basis since we can always change the basis and obtain a similar matrix. Hence it is well-defined.