Lecture 9: More about diagonalization

Example 1: (Benefit of diagonalization)

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & 2 \\ -4 & -5 & 5 \end{pmatrix}$. Show that A is diagonalizable and compute A^n.

Char. poly. of A: $f(t) = -(t-1)(t-3)(t-5)$

$\therefore f(t)$ splits $\Rightarrow$ ① holds.

② automatically holds since multiplicity of each eigenvalue is 1.

$\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 5$ are eigenvalues.

$\therefore A$ is diagonalizable.

Now, to diagonalize A, we need to find basis for $E_{\lambda_1}, E_{\lambda_2}$ and E_{λ_3}.

Simple calculation give:

$$E_{\lambda_1} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\},\ E_{\lambda_2} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right\},\ E_{\lambda_3} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

$\therefore \beta = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis of $\mathbb{R}^3$ diagonalizing A.

Thus, $D := \begin{pmatrix} 1 & & 0 \\ & 3 & \\ 0 & & 5 \end{pmatrix} = Q^{-1} A Q$ where $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\therefore A^n = \underbrace{(QDQ^{-1}) \ldots (QDQ^{-1})}_{n} = QD^nQ^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1^n & & \\ & 3^n & \\ & & 5^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3^n - 5^n & 3^n & 5^n - 3^n \\ 1 - 5^n & 0 & 5^n \end{pmatrix}$$

Eigenspace and Direct Sum

Definition 1: Let $W_1, W_2, \dots, W_k$ be subspaces of V.

The sum of subspaces is the set:

$$\{\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_k : \vec{v}_i \in W_i \text{ for } 1 \le i \le k\}$$

Denote it as: $W_1 + W_2 + \dots + W_k$ or $\sum_{i=1}^{k} W_i$.

($W_1 + \dots + W_k$ is a subspace of V)

Example 2: $\mathbb{R}^4 = W_1 + W_2$ where

$W_1 = \{(a,b,c,0) : a,b,c \in \mathbb{R}\}$; $W_2 = \{(0,a,b,c) : a,b,c \in \mathbb{R}\}$

(For each $(a,b,c,d) \in \mathbb{R}^4$, $(a,b,c,d) = \underbrace{\underbrace{(a,b,c,0)}_{\vec{w}_1} + \underbrace{(0,0,0,d)}_{\vec{w}_2}}_{\in W_1 + W_2}$)

The decomposition is NOT unique.

Not Good! We need a unique decomposition.

Definition 2: Let $W_1, W_2, \dots, W_k$ = subspaces of V. We say V is a <u>direct sum</u> of $W_1, \dots, W_k$ if:

① $V = \sum_{i=1}^{k} W_i$ ② $W_j \cap \sum_{i \ne j} W_i = \{\vec{0}\}$ for each j.

Denote it as: $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$

Example 3: $W_1 = \{(a,b,0,0) : a,b \in \mathbb{R}\}$, $W_2 = \{(0,0,c,0) : c \in \mathbb{R}\}$

$W_3 = \{(0,0,0,d) : d \in \mathbb{R}\}$

Then: $\mathbb{R}^4 = W_1 + W_2 + W_3$ $\left((a,b,c,d) = \underbrace{(a,b,0,0)}_{\in W_1} + \underbrace{(0,0,c,0)}_{\in W_2} + \underbrace{(0,0,0,d)}_{\in W_3}\right)$

Now, easy to check: $\underbrace{W_1 \cap (W_2+W_3) = W_2 \cap (W_1+W_3) = W_3 \cap (W_1+W_2)}_{= \{\vec{0}\}}$

$\therefore \mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$.

<u>Theorem 1</u>: Let $W_1, W_2, \ldots, W_k$ be subspaces.
The following are equivalent:

(a) $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$

(b) $V = \sum_{i=1}^{k} W_i$ and $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k = 0 \Rightarrow \vec{v}_1 = \vec{v}_2 = \ldots = \vec{v}_k = \vec{0}$.

(c) For each $\vec{v} \in V$, $\vec{v}$ can be uniquely written as:

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k \quad (\vec{v}_i \in W_i)$$

(d) If γ_i = ordered basis for W_i, then:
$\gamma_1 \dot{\cup} \gamma_2 \dot{\cup} \ldots \dot{\cup} \gamma_k$ is an ordered basis for V.

(e) For each $i = 1, 2, \ldots, k$, $\exists$ ordered basis γ_i for W_i such that:
$\gamma_1 \dot{\cup} \gamma_2 \dot{\cup} \ldots \dot{\cup} \gamma_k$ is an ordered basis for V.

<u>Proof</u>: Later.

<u>Theorem 2</u>: $T: V \to V$ (finite-dim). V is diagonalizable if and only if V is a direct sum of eigenspaces of T.

<u>Proof</u>: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ = distinct eigenvalues of T.

($\Rightarrow$) T = diagonalizable. Then $\exists \gamma_i$ = ordered basis for $E_{\lambda_i} \Rightarrow$
$\gamma_1 \dot{\cup} \gamma_2 \dot{\cup} \ldots \dot{\cup} \gamma_k$ = ordered basis of V.
From Thm 1 (e), $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \ldots \oplus E_{\lambda_k}$.

($\Leftarrow$) Let $V = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k}$. Choose γ_i be basis of E_{λ_i}. Then:

by Thm 1(d), $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k$ is a basis for V consisting of eigenvectors. So, T is diagonalizable.

<u>Example</u>: Let a, b, c be distinct. $(a, b, c \neq 0)$

Define $T: \mathbb{R}^n \to \mathbb{R}^n$ by:

$$T(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_{n-2}, bx_{n-1}, cx_n)$$

Let β: standard ordered basis. Then:

$$[T]_\beta = \begin{pmatrix} a & & & & \\ & \ddots & & & \\ & & a & & \\ & & & b & \\ & & & & c \end{pmatrix}. \quad \therefore T \text{ is diagonalizable with eigenvalues}$$

$$\lambda_1 = a, \lambda_2 = b, \lambda_3 = c.$$

Easy to check that:

$$E_{\lambda_1} = \{(x_1, x_2, \ldots, x_{n-2}, 0, 0) : x_1, x_2, \ldots, x_{n-2} \in \mathbb{R}\}$$

$$E_{\lambda_2} = \{(0, 0, \ldots, 0, x_{n-1}, 0) : x_{n-1} \in \mathbb{R}\}$$

$$E_{\lambda_3} = \{(0, 0, \ldots, 0, x_n) : x_n \in \mathbb{R}\}$$

Clearly: $\mathbb{R}^n = E_{\lambda_1} + E_{\lambda_2} + E_{\lambda_3}$ and

$$E_{\lambda_1} \cap (E_{\lambda_2} + E_{\lambda_3}) = E_{\lambda_2} \cap (E_{\lambda_1} + E_{\lambda_3}) = E_{\lambda_3} \cap (E_{\lambda_1} + E_{\lambda_2}) = \{\vec{0}\}$$

$$\therefore \mathbb{R}^n = E_{\lambda_1} \oplus E_{\lambda_2} \oplus E_{\lambda_3}$$

(Diagonalizable $\Leftrightarrow$ Direct sum of eigenspaces E_{λ_i}'s)