Lecture 5: More about eigenvalues and eigenvectors

Definition 1: Let $T: V \to V$ (V = finite-dim). Let β = ordered basis of V. The characteristic polynomial of T is defined as:

$$f(t) = \det(A - tI_n) \text{ where } A = [T]_\beta$$

Question: Is this definition well-defined?
That is, if we choose different ordered basis β', would $f(t)$ be the same?

Answer: Yes. Recall that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, $Q = [I_V]_{\beta'}^{\beta}$.

So,
$$\det([T]_{\beta'} - tI_n) = \det(Q^{-1}[T]_\beta Q - tI_n)$$
$$= \det(Q^{-1}([T]_\beta - tI_n)Q)$$
$$= \underbrace{\det(Q^{-1})}_{= 1/\det(Q)} \det([T]_\beta - tI_n)\det(Q)$$
$$= \det([T]_\beta - tI_n)$$

$\therefore$ $f(t)$ is independent of the chosen β.

Remark: We will show that λ is an eigenvalue of T iff $T - \lambda I$ is not invertible iff $\det([T]_\beta - \lambda I) = 0$ iff $f(t) = 0$

Hence, understanding $f(t)$ is important.

Using $f(t)$, eigenvalue/eigenvector of T can be computed.

Example 1: $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by:

$$T(f(x)) = f(x) + (x+1)f'(x)$$

Let $\beta = \{1, x, x^2\}$ be an ordered basis of $P_2(\mathbb{R})$.

Then: $T(1) = 1$; $T(x) = 2x+1$; $T(x^2) = 3x^2 + 2x$

$$\therefore [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

The characteristic polynomial of T:

$$f(t) = \det([T]_\beta - tI) = \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = (1-t)(2-t)(3-t)$$

Thus, $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ are eigenvalues of T since $f(\lambda_i) = 0$ for $i = 1, 2, 3$.

Solving $[T]_\beta \vec{v} = \lambda_1 \vec{v}$, we get $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a sol.

Hence, $1 \in P_2(\mathbb{R})$ is an eigenvector of T with eigenvalue $\lambda_1 = 1$

(Reason: Let $f(x) = 1 \in P_2(\mathbb{R})$. Then $[f]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$[T]_\beta \vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow [T]_\beta [f]_\beta = \lambda_1 [f]_\beta$$
$$\Rightarrow [Tf]_\beta = [\lambda_1 f]_\beta \Rightarrow Tf = \lambda_1 f \text{)}$$

Similarly, we can find eigenvectors $\vec{v}_2$ and $\vec{v}_3$ with eigenvalues λ_2 and λ_3 respectively. (Exercise)

Relationship between eigenvectors of T and $[T]_\beta$

We have seen that we can compute eigenvector of $T: V \to V$ by considering its matrix representation.

A rigorous explanation

Consider:
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & V \arrow[d, "\phi_\beta"] \\ \mathbb{F}^n \arrow[r, "L_A"', "A=[T]_\beta"] & \mathbb{F}^n \end{tikzcd}$$
; $\beta =$ ordered basis ; $\phi_\beta(\vec{v}) = [\vec{v}]_\beta$

① Suppose $T(\vec{v}) = \lambda \vec{v}$ with $\vec{v} \neq \vec{0}$.

Then: $A\phi_\beta(\vec{v}) = L_A \circ \phi_\beta(\vec{v}) = \phi_\beta \circ T(\vec{v}) = \phi_\beta(\lambda \vec{v})$
$$= \lambda \phi_\beta(\vec{v})$$

$\therefore$ λ is an eigenvalue of A ; $\phi_\beta(\vec{v})$ is an eigenvector associated with λ.

② Now, suppose $A\vec{x} = \lambda \vec{x}$, $\vec{x} \neq \vec{0}$. Let $\vec{v} = \phi_\beta^{-1}(\vec{x}) \in V$

Then: $\phi_\beta(T(\vec{v})) = L_A \circ \phi_\beta(\vec{v}) = L_A(\vec{x}) = A\vec{x} = \lambda \vec{x}$
$$= \phi_\beta(\lambda \vec{v})$$

Thus, $\phi_\beta(T(\vec{v})) = \phi_\beta(\lambda \vec{v})$

Since ϕ_β is an isomorphism, $T(\vec{v}) = \lambda \vec{v}$.

Theorem 1: $\vec{v}$ = eigenvector of T with eigenvalue λ

$$\Updownarrow$$

$\phi_\beta(\vec{v})$ = eigenvector of A with eigenvalue λ.

Theorem 2: Let $T: V \to V$; β = ordered basis of V.

λ is an eigenvalue of T iff $\det([T]_\beta - \lambda I) = 0$

Proof: From the above conclusion, λ is an eigenvalue of T iff λ is an eigenvalue of $A := [T]_\beta$.
iff $\det(A - \lambda I) = 0$

Remark: To find eigenvalues/eigenvector of $T: V \to V$:

Step 1: Compute matrix representation of $T = [T]_\beta$

Step 2: Compute eigenvalues λ of $[T]_\beta$ by solving characteristic polynomial. Find eigenvector $\vec{x} \in \mathbb{F}^n$ by solving $([T]_\beta - \lambda I)\vec{x} = \vec{0}$.

Step 3: Obtain eigenvector of T: $\vec{v} = \phi_\beta^{-1}(\vec{x})$

λ is also an eigenvalue of T.

Example 2: Let $V = \{ a\sin x + b\cos x + ce^x : a, b, c \in \mathbb{R} \}$.

Then, V is a vector space over $\mathbb{R}$.

Consider $\beta = \{\sin x, \cos x, e^x\}$. Obviously, β is a basis of V.

Define $T: V \to V$ by:

$$T(a\sin x + b\cos x + ce^x) = (a+2b)\sin x + (6b+4c)\cos x + 11c\,e^x \in V$$

To find eigenvectors/eigenvalues of T:

Step 1: Compute matrix representation

$$A := [T]_\beta = \begin{pmatrix} | & | & | \\ [T(\sin x)]_\beta & [T(\cos x)]_\beta & [T(e^x)]_\beta \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 6 & 4 \\ 0 & 0 & 11 \end{pmatrix}$$

Step 2: Solve characteristic polynomial:

$$f(t) := \det(A - \lambda I) = 0 \iff (t-1)(t-6)(t-11) = 0$$

$\therefore$ Eigenvalues of A: $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 11$.

Find eigenvectors: for λ_1, solve $(A - 1I)\vec{x} = \vec{0}$

$N(A - 1I) = \{ t\begin{pmatrix}1\\0\\0\end{pmatrix} : t \in \mathbb{R} \}$. Pick $\vec{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

for λ_2, $N(A - 6I) = \{ t\begin{pmatrix}2\\5\\0\end{pmatrix} : t \in \mathbb{R} \}$. Pick $\vec{x}_2 = \begin{pmatrix}2\\5\\0\end{pmatrix}$

for λ_3, $N(A - 11I) = \{ t\begin{pmatrix}4\\20\\25\end{pmatrix} : t \in \mathbb{R} \}$. Pick $\vec{x}_3 = \begin{pmatrix}4\\20\\25\end{pmatrix}$.

Step 3: Find eigenvectors of T by $\vec{v} = \phi_\beta^{-1}(\vec{x})$.

$\vec{v}_1 = \phi_\beta^{-1}(\vec{x}_1) = 1\sin x$

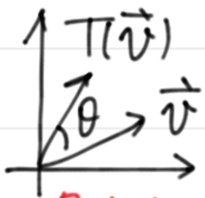
$\vec{v}_2 = \phi_\beta^{-1}(\vec{x}_2) = 2\sin x + 5\cos x$

$\vec{v}_3 = \phi_\beta^{-1}(\vec{x}_3) = 4\sin x + 20\cos x + 25e^x$

Eigenvectors of T w.r.t. $\lambda_1 = 1$, $\lambda_2 = 6$ and $\lambda_3 = 11$ resp.

Example 3: Last time: $T: \mathbb{R}^2 \to \mathbb{R}^2$ where:

$$T(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$$



$$[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotate counter-clockwisely by θ standard ordered basis

To compute eigenvalues, consider:

$$f(t) = \det([T]_\beta - tI) = 0 \Leftrightarrow \det\begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} = 0$$

$$\Leftrightarrow t^2 - 2t\cos\theta + 1 = 0$$

$$\Leftrightarrow t = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

But $4\cos^2\theta - 4 < 0$ if $0 < \theta < \pi$

Thus, T doesn't have eigenvalues/eigenvectors if $0 < \theta < \pi$.

Theorem 3: Let $A \in M_{n\times n}(\mathbb{F})$. Then:

1. The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.
2. A has at most n distinct eigenvalues.
   (degree n polynomial has at most n roots)

Proof: Use induction to prove that:

$$f(t) = (A_{11}-t)(A_{22}-t)\dots(A_{nn}-t) + \underbrace{g(t)}_{\deg \leq n-2}$$

(Homework)

Example 4: Let $C^\infty(\mathbb{R}) = \{$collection of all smooth functions $f: \mathbb{R} \to \mathbb{R}\}$

Define: $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f) = \frac{d^2 f}{dx^2}$

infinite dimensional

Let $n = 0, 1, 2, \dots$

Consider $f_n(x) = \sin nx$.

Then: $T(f_n(x)) = -n^2 \sin nx = -n^2 f_n(x)$

So, $-n^2$ is an eigenvalue of T, $\sin nx$ is an eigenvector.

Since $n = 0, 1, 2, \dots$, T has infinitely many eigenvalues.

($f_n(x)$ is also called an <u>eigenfunction</u>.)

Theorem 4: Let $T: V \to V$ and λ = eigenvalue.

can be infinite-dim

Then: $\vec{v}$ is an eigenvector associated to $\lambda \iff$

$$\vec{v} \neq 0 \text{ and } \vec{v} \in N(T - \lambda I)$$

Proof: $\vec{v}$ = eigenvector of $\lambda \iff T(\vec{v}) = \lambda \vec{v}$ and $\vec{v} \neq \vec{0}$

$\iff (T - \lambda I)\vec{v} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$

$\iff \vec{v} \neq \vec{0}$ and $\vec{v} \in N(T - \lambda I)$