Lecture 4: Eigenvalues and Eigenvectors

Recall: • Let $T: V \to V$ (β = ordered basis of V $= \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$)

Then, the matrix representation:

$$[T]_\beta = \begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \\ | & & | \end{pmatrix}$$

• Also, $T(\vec{v}_j) = \sum_{i=1}^{n} A_{ij} \vec{v}_i$

• (Change of coordinates). Let β and β' be two ordered basis of V. Let $Q = [I_V]_{\beta'}^{\beta}$.

Then: $[T]_{\beta'} = Q^{-1} [T]_\beta Q$

Goal: Given $T: V \to V$, find an ordered basis β such that

$$[T]_\beta = \begin{pmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{pmatrix} := D = \text{diagonal matrix}$$

Reason: Easy to manipulate.

① $\det(D) = d_{11} d_{22} \dots d_{nn}$

② $D^{-1} = \begin{pmatrix} d_{11}^{-1} & & & \\ & d_{22}^{-1} & & \\ & & \ddots & \\ & & & d_{nn}^{-1} \end{pmatrix}$ if $\det(D) \neq 0$

and many other advantages.

We will answer 2 questions:

① Does there exist such β to diagonalize T?

② If yes, how to find β?

$\Downarrow$

$\boxed{\text{Concept of eigenvalues/eigenvectors}}$

Observation: Let $T: V \to V$ (V = finite-dim)

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ = ordered basis.

Suppose $[T]_\beta = \begin{pmatrix} d_{11} & & 0 \\ & d_{22} & \\ 0 & & \ddots & d_{nn} \end{pmatrix} \leftarrow$ Diagonal matrix

Then: $T(\vec{v}_j) = \sum_{i=1}^{n} D_{ij} \vec{v}_i = d_{jj} \vec{v}_j$ ($\because d_{ij} = 0$ if $i \neq j$)

$= \lambda_j \vec{v}_j$ where $\lambda_j = d_{jj}$

Conversely, if $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an ordered basis such that $T(\vec{v}_j) = \lambda_j \vec{v}_j$.

Then: $[T(\vec{v}_j)]_\beta = \begin{pmatrix} 0 \\ \vdots \\ \lambda_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th}$

Hence, $[T]_\beta = \begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \\ | & & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{pmatrix} \leftarrow$ Diagonal matrix

Remark: From above observation, diagonalizing T is related to finding $\boxed{T(\vec{v}) = \lambda \vec{v}}$

Definition 1: A linear operator $T: V \to V$ on a finite-dim vector space V is called diagonalizable if $\exists$ ordered basis β such that $[T]_\beta$ is a diagonal matrix.

A square matrix A is called diagonalizable if L_A is diagonalizable. (Recall: $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is defined as $L_A(\vec{v}) = A\vec{v}$)

Definition 2: Let $T: V \to V$ (V = finite-dim). A non-zero $\vec{v} \in V$ is called an eigenvector of T if $\exists$ some scalar $\lambda \in \mathbb{F}$ such that $T(\vec{v}) = \lambda \vec{v}$. λ is called the eigenvalue of the eigenvector $\vec{v}$.

Let $A \in M_{n \times n}(\mathbb{F})$. $\vec{v} \in \mathbb{F}^n$ is an eigenvector of A if $\vec{v}$ is an eigenvector of L_A.
(That is, $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in \mathbb{F}$ and λ is called the eigenvalue of $\vec{v}$)

Remark:
- Eigenvector CANNOT be $\vec{0}$.
- Eigenvalue CAN be 0.

From our observation, we have the following theorem:

Theorem 1: A linear operator $T: V \to V$ (V = finite-dim) is diagonalizable iff $\exists$ ordered basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ consisting of eigenvectors of T.
Also, if T is diagonalizable using the ordered basis β, then: $[T]_\beta = \begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{pmatrix}$ where $T(\vec{v}_i) = d_{ii}\vec{v}_i$

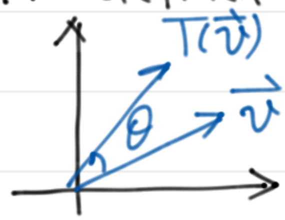
Example 1: Let $A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $\vec{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Then: $L_A(\vec{v}_1) = A\vec{v}_1 = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$L_A(\vec{v}_2) = A\vec{v}_2 = -1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; \quad L_A(\vec{v}_3) = A\vec{v}_3 = -3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

So, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are eigenvectors of A with eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = -3$ respectively.

Example 2: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:



(Rotate $\vec{v}$ counter-clockwisely by θ)

If $0 < \theta < \pi$, then $\vec{v}$ and $T(\vec{v})$ are not collinear.
So, $T(\vec{v}) \neq \lambda\vec{v}$ for any $\lambda \in \mathbb{R}$.
T has no eigenvectors or eigenvalues.

Remark: Some $T: V \to V$ (or $A \in M_{n\times n}(\mathbb{F})$) don't have eigenvalues or eigenvectors.

First question: How to determine whether eigenvalues/eigenvectors exist?

Theorem 2: Let $A \in M_{n\times n}(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is an eigenvalue of A iff $\det(A-\lambda I_n)=0$.

Proof: $\lambda =$ eigenvalue of $A \Leftrightarrow \exists \vec{v} \neq 0 \in V \ni A\vec{v} = \lambda \vec{v}$

$\Leftrightarrow (A-\lambda I)\vec{v} = \vec{0}$ for some $\vec{v} \neq 0 \in V$

$\Leftrightarrow A-\lambda I$ is not invertible

(if $A-\lambda I$ is invertible, then:
$(A-\lambda I)\vec{v} = \vec{0} \Rightarrow \vec{v} = (A-\lambda I)^{-1}\vec{0} = \vec{0}$.
Contradiction)

$\Leftrightarrow \det(A-\lambda I) = 0$

Definition 3: Let $A \in M_{n\times n}(\mathbb{F})$. The polynomial $f(t) = \det(A - tI_n)$ is called the <u>characteristic polynomial</u> of A.

Remark: A has an eigenvalue iff the characteristic polynomial of A has roots.

Finding eigenvalues/eigenvectors of $A \in M_{n\times n}(\mathbb{F})$

Example 3: Find eigenvalues/eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Step 1: Find $f(t) = \det(A - tI) = \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)$

$$= \det\begin{pmatrix} 2-t & 1 \\ 1 & 2-t \end{pmatrix} = (2-t)^2 - 1$$

$$= (t-1)(t-3)$$

Step 2: Solve $f(t) = 0$. $f(t) = 0 \Leftrightarrow t = 1$ or $t = 3$.

$\therefore \lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues of A.

Step 3: Solve $A\vec{v} = \lambda_1 \vec{v}$ and $A\vec{v} = \lambda_2 \vec{v}$.

For $\lambda_1 = 1$, solve $A\vec{v} = \vec{v} \Leftrightarrow (A - I)\vec{v} = \vec{0}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a sol.

For $\lambda_2 = 3$, solve $A\vec{v} = 3\vec{v}$. $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a sol.

Hence, $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with eigenvalues λ_1 and λ_2 respectively.

Example 4: Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. (Rotate counter-clockwisely by $\pi/2$)

Consider $f(t) = \det(A - tI) = \det\begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1$

Note that $f(t) \neq 0$ for all $t \in \mathbb{R}$.

Thus, A has no eigenvalue (and hence no eigenvector)

<u>Definition 3:</u> Let $T: V \to V$ (V = finite-dim), β = ordered basis. Define the characteristic polynomial $f(t)$ of T as:

$f(t) = \det(A - tI_n)$ where $A = [T]_\beta$

(Next time :- prove that $f(t)$ is independent of the chosen β

- λ is an eigenvalue of T iff $f(\lambda) = 0$

[Hence, eigenvalue/eigenvector can be computed from $f(t)$ as before.])