Lecture 26: Jordan Canonical Form (Part 4)

Claim 8: Suppose β = basis of V = disjoint union of cycles. Then:

① For each cycle γ in β, $W = \text{span}(\gamma)$ is T-invariant and $[T_W]_\gamma$ = Jordan block.

② β = JC basis for V.

Proof: Let $\gamma = \{\underbrace{(T-\lambda I)^{p-1}x}_{=\vec{v}_1}, \dots, \underbrace{(T-\lambda I)x}_{=\vec{v}_{p-1}}, x\}$

$\therefore (T-\lambda I)\vec{v}_1 = \vec{0} \Rightarrow T\vec{v}_1 = \lambda\vec{v}_1 \in W$

For $i > 1$, $(T-\lambda I)\vec{v}_i = \vec{v}_{i-1} \Rightarrow T(\vec{v}_i) = \lambda\vec{v}_i + \vec{v}_{i-1} \in W$

$\therefore W$ is T-invariant and $[T]_\gamma = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$

② is the repeated argument of ①.

Claim 9: Let λ = eigenvalue of T. K_λ has a basis β = union of disjoint cycles w.r.t. λ.

Proof: By M.I. on $n = \dim(K_\lambda)$.

When $n = 1$, trivial.

Suppose the result is true for $\dim(K_\lambda) < n$.

When $\dim(K_\lambda) = n$. Let $U = (T-\lambda I)|_{K_\lambda}$.

Then: $\dim(R(U)) < \dim(K_\lambda)$ because:

$$\dim(K_\lambda) = \dim(N(U)) + \dim(R(U)) \geq 1 + \dim(R(U))$$

(since $N(U) \supseteq E_\lambda$ and $\dim(E_\lambda) \geq 1$)

Let K_λ' = generalized eigenspace corresponding to λ of $T|_{R(U)}$

(clearly $R(U)$ is T-invariant)

Easy to check that $R(U) = K_\lambda'$. (Check!)

By induction hypothesis, $\exists$ disjoint cycles $\gamma_1, \gamma_2, \dots, \gamma_q$ of $T|_{R(U)}$ $\ni$

$\gamma = \bigcup_{i=1}^{q} \gamma_i$ is a basis for $R(U)$ $(= K'_\lambda)$

Let $\gamma_i = \{(T|_{R(U)} - \lambda I)^{m_i} x_i, \dots, \underset{\in R(U)}{x_i}\}$

Now, let $x_i = U v_i = (T-\lambda I) v_i$, $v_i \in K_\lambda$.

$\therefore \gamma_i = \{(T-\lambda I)^{m_i+1} v_i, \dots, (T-\lambda I) v_i\}$

Define: $\tilde{\gamma}_i = \{\underbrace{(T-\lambda I)^{m_i+1} v_i}_{= w_i}, \dots, (T-\lambda I) v_i, v_i\}$

Note that $\bigcup_{i=1}^{q} \gamma_i$ is lin ind., so,

$S = \{w_1, \dots, w_q\}$ are lin ind. subset of E_λ.

Extend S to a basis for E_λ: $\{w_1, w_2, \dots, w_q, u_1, \dots, u_s\}$

Then: $\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_q, \{u_1\}, \dots, \{u_s\}$ are disjoint union of cycles (w.r.t. λ) $\ni$ initial vectors are lin. ind.

$\therefore \tilde{\gamma} = \bigcup_{i=1}^{q} \tilde{\gamma}_i \cup \{u_1\} \cup \dots \cup \{u_s\}$ is lin. ind. subset of K_λ.

Now, we show that $\tilde{\gamma}$ is a basis for K_λ.

Suppose $|\gamma| = r = \dim(R(U))$

basis of $R(U)$

Then: $|\tilde{\gamma}| = r + q + s$.

$\because \{w_1, \dots, w_q, u_1, \dots, u_s\}$ is a basis for $E_\lambda = N(U) = N((T-\lambda I)|_{K_\lambda})$

$\therefore \dim(N(U)) = q + s$.

$\therefore \dim(K_\lambda) = \dim(R(U)) + \dim(N(U)) = r + q + s = |\tilde{\gamma}|$.

$\therefore \tilde{\gamma}$ is a basis for K_λ.

<u>Claim 10:</u> T has JCF.

<u>Proof:</u> Let $\lambda_1, \dots, \lambda_k$ = distinct eigenvalues of T.

For each i, $\exists$ basis of K_{λ_i} which is disjoint union of cycles.

Let $\beta = \beta_1 \cup \dots \cup \beta_k$. Then, β = basis of V and β = disjoint union of cycles. Hence, β = JC basis.

<u>Formula for dot diagram</u> : Let β_i = basis of K_{λ_i}

<u>Claim 11:</u> Let r = positive integer. The vectors in β_i in the first r rows of the dot diagram is a basis for $N((T - \lambda_i I)^r)$.

$\left(\therefore \text{ \# of dots in the first } r \text{ rows of dot diagram} = \text{Nullity}((T - \lambda_i I)^r) \right)$

<u>Proof:</u> Clearly, $N((T - \lambda_i I)^r) \subseteq K_{\lambda_i}$

Let $U = (T - \lambda_i I)^r|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$

Then: $N(U) = N((T - \lambda_i I)^r)$

Let $S_1 = \{x \in \beta_i : U(x) = 0\}$ and $S_2 = \{x \in \beta_i : U(x) \neq 0\}$

Let $m_i = \dim(K_{\lambda_i})$. Then: $|S_1| + |S_2| = m_i$.

Note that for $\forall x \in \beta_i$, $x \in S_1 \Leftrightarrow x$ is the first r rows in the dot diagram.

$\therefore |S_1| = \#$ of dots in the 1st r rows.

For $x \in S_2$, Ux is exactly the dot r places up its col.

$\therefore U$ map S_2 1-1 into β_i

$\therefore \{U(x) : x \in S_2\}$ = basis for $R(U)$.

$\therefore \dim(R(U)) = |\{U(x) : x \in S_2\}| = |S_2|$.

Thus, $\dim(N(U)) = \dim(K_{\lambda_i}) - \dim(R(U)) = m_i - |S_2| = |S_1|$

But S_1 is lin. ind. subset of $N(U)$. $\therefore S_1$ = basis for $N(U)$.

Claim 12: # of Jordan block corresponding to λ_i

$$= \# \text{ of dots in the 1st row} = \dim(E_{\lambda_i})$$

Claim 13: ① $r_1 = \dim(V) - \operatorname{rank}(T - \lambda_i I)$

② $r_j = \operatorname{rank}((T-\lambda_i I)^{j-1}) - \operatorname{rank}((T-\lambda_i I)^j)$

Proof: By Claim 11, $r_1 + r_2 + \dots + r_j = \dim(N((T-\lambda_i I)^j))$

$$= \dim(V) - \operatorname{rank}((T-\lambda_i I)^j)$$

$\therefore r_1 = \dim(V) - \operatorname{rank}((T - \lambda_i I))$

Now, $r_j = (r_1 + \dots + r_j) - (r_1 + \dots + r_{j-1})$

$$= (\dim(V) - \operatorname{rank}((T-\lambda_i I)^j)) - (\dim(V) - \operatorname{rank}((T-\lambda_i I)^{j-1}))$$

$$= \operatorname{rank}((T-\lambda_i I)^{j-1}) - \operatorname{rank}((T-\lambda_i I)^j).$$