Lecture 23: Jordan Canonical Form (Part I)

Recall: If char poly of T splits and $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$, then: T is diagonalizable.

Also, $1 \leq \dim(E_{\lambda_i}) \leq m_i$ = multiplicity

Hence, E_{λ_i} has to be large enough.

How about if $V \underset{\neq}{\supseteq} E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$?

Goal: Find matrix representation of T < simple / similar to diagonal matrix.

Specifically: Find β such that:

$$[T]_\beta = \begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_k \end{pmatrix}$$

A_i = block square matrix

Block diagonal

$$A_i = \begin{pmatrix} \lambda & 1 & & O \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda \end{pmatrix}$$ where λ is an eigenvalue of T.

Definition: $[T]_\beta$ is called the <u>Jordan canonical form</u> of T

A_i is called a <u>Jordan block</u> corresponding to λ.

β is called the <u>Jordan canonical basis</u>.

Example 1: Let $\beta = \{\vec{v}_1, \dots, \vec{v}_6\}$ such that:

$$[T]_\beta = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & \\ & & \boxed{\lambda_2} \end{pmatrix}$$

λ_1, λ_2 = eigenvalues with multiplicity = 3

Easy to check: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(T-\lambda_1 I)^2 \vec{v}_3, (T-\lambda_1 I)\vec{v}_3, \vec{v}_3\}$

$\{\vec{v}_4, \vec{v}_5\} = \{(T-\lambda_2 I)\vec{v}_5, \vec{v}_5\}$

$\{\vec{v}_6\} = \{\vec{v}_6\}$

Remark: - JCF is NOT unique

- $(T-\lambda_1 I)^3 \vec{v}_3 = \vec{0}$, $(T-\lambda_2 I)^2 \vec{v}_5 = \vec{0}$; $(T-\lambda_2 I)\vec{v}_6 = \vec{0}$

Definition 1: Let $T: V \to V$ and $\lambda \in \mathbb{F}$. $\vec{x} \in V$ $(\vec{x} \neq \vec{0})$ is called a generalized eigenvector w.r.t. λ if $(T-\lambda I)^p \vec{x} = \vec{0}$ for some positive $p \in \mathbb{N}$.

Definition 2: Generalized eigenspace of T w.r.t. λ:

$$K_\lambda = \{\vec{x} \in V : (T-\lambda I)^p(\vec{x}) = \vec{0} \text{ for some positive } p \in \mathbb{N}\}$$

Definition 3: Let $\vec{x} \in K_\lambda$ and $\vec{x} \neq \vec{0}$. Suppose p is the smallest integer such that $(T-\lambda I)^p(\vec{x}) = \vec{0}$. Then:

$\{(T-\lambda I)^{p-1}(\vec{x}), (T-\lambda I)^{p-2}(\vec{x}), \dots, \vec{x}\}$ is called a cycle of generalized eigenvectors of T corresponding to λ.

$\vec{x}$ is called end vector.

$(T-\lambda I)^{p-1}(\vec{x})$ is called the initial vector.

<u>Observation:</u> Let β = basis = cycle $= \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$= \{(T-\lambda I)^{n-1}(\vec{x}), \ldots, \vec{x}\}$.

Then: $(T-\lambda I)\vec{v}_1 = \vec{0} \Rightarrow T(\vec{v}_1) = \lambda\vec{v}_1$

$(T-\lambda I)(\vec{v}_2) = \vec{v}_1 \Rightarrow T(\vec{v}_2) = \lambda\vec{v}_2 + \vec{v}_1$

$\vdots$

$(T-\lambda I)(\vec{v}_n) = \vec{v}_{n-1} \Rightarrow T(\vec{v}_n) = \lambda\vec{v}_n + \vec{v}_{n-1}$

$\therefore [T]_\beta = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$ = Jordan block!

Goal: Find basis of V = disjoint union of cycles!

<u>Properties of K_λ</u> (Proof: later!)

① K_λ is T-invariant: Let $\vec{x} \in K_\lambda$. $(T-\lambda I)^p(\vec{x}) = \vec{0}$

Now, $(T-\lambda I)^p(T(\vec{x})) = T\underbrace{((T-\lambda I)^p(\vec{x}))}_{\vec{0}} = \vec{0}$

$\therefore T(\vec{x}) \in K_\lambda$.

② $E_\lambda \subseteq K_\lambda$ (trivial)

③ Let $T: V \to V$ (fin-dim). Suppose char poly of T splits. Let $\lambda_1, \ldots, \lambda_k$ be (all) distinct eigenvalues. Then:

- $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$.
- $\dim(K_{\lambda_i}) = m_i$ = multiplicity of λ_i and $K_{\lambda_i} = N((T-\lambda_i I)^{m_i})$
- T is diagonalizable iff $E_{\lambda_i} = K_{\lambda_i}$ for all i.

(T = diagonalizable $\Leftrightarrow \dim(E_{\lambda_i}) = m_i \ \forall i$

$\Leftrightarrow \dim(E_{\lambda_i}) = \dim(K_{\lambda_i}) \Leftrightarrow E_{\lambda_i} = K_{\lambda_i}$

($\because E_{\lambda_i} \subseteq K_{\lambda_i}$))

④ If $\beta = \gamma_1 \dot\cup \gamma_2 \dot\cup \cdots \dot\cup \gamma_k$ (γ_i = cycles) is a basis of V, then β is Jordan canonical basis.

(5) Let $\gamma_1 = \{(T-\lambda_1 I)^{p_1 - 1}(\vec{v}_1), \dots, \vec{v}_1\}$
$\vdots$
$\gamma_q = \{(T-\lambda_q I)^{p_q - 1}(\vec{v}_q), \dots, \vec{v}_q\}$
cycles.

If initial vectors are linearly independent, then $\gamma_1, \dots, \gamma_q$ are disjoint and $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_q$ is linearly independent.

(6) Put $q=1$. We get: every cycle is linearly independent.

(7) (Main Thm) Let $T: V \to V$ (fin-dim). Let λ = eigenvalue.
Then K_λ has a basis consisting of disjoint union of cycle of generalized eigenvectors of λ.

HENCE, if char poly of T splits, then T has a Jordan canonical form.

(Let $\lambda_1, \dots, \lambda_k$ = distinct eigenvalues. Let β_i = basis of K_{λ_i}
= disjoint union of cycles.

Then: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ = basis of V.
= disjoint union of cycles.

$\therefore [T]_\beta$ = JCF by (4))

<u>Computation of JCF</u>:

<u>Example 2</u>: Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 4 & 0 \\ -2 & -3 & 6 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$. Find JCF.

<u>Solution</u>: Char poly $= (6-t)(2-t)^2 \therefore \lambda_1 = 6, \lambda_2 = 2$ are eigenvalues.

Now, $\dim(K_{\lambda_1}) = m_1 = 1$, $\dim(K_{\lambda_2}) = 2 = m_2$

And $K_{\lambda_1} = N(T-6I) = E_{\lambda_1}$; $K_{\lambda_2} = N((T-2I)^2)$

Goal: Find basis β_i for K_{λ_i} which is union of cycles.

For $\lambda_1 = 6$, β_1 has one vector. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ = eigenvector of $\lambda_1 = 6$.

$\therefore$ let $\beta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$, $\dim(K_{\lambda_2}) = 2$. $\therefore \beta_2$ has length $= 2$. It can be:

① union of two cycles of length 1.

(Impossible: ① implies there are two linearly independent eigenvectors of $\lambda_2 = 2$. But we can check $\dim(E_{\lambda_2}) = 1$. $\therefore$ ① is impossible)

② one cycle of length 2.

$\therefore$ Need to find: $\beta_2 = \{(A - 2I)\vec{v}, \vec{v}\}$ = basis of K_{λ_2}.

$\therefore$ Find $\underset{\in K_{\lambda_2}}{\vec{v} \neq \vec{0}} \ni (A - 2I)(\vec{v}) \neq 0$ but $(A - 2I)^2(\vec{v}) = \vec{0}$.

We can check that $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in K_{\lambda_2} = N((A - 2I)^2)$, $(A - 2I)\vec{v} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \neq \vec{0}$

Hence, $\beta_2 = \left\{ \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\beta = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ = Jordan canonical basis.

and $[T]_\beta = \begin{pmatrix} 6 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$.