Lecture 13: More about inner product

Definition 1: (Frobenius inner product) Let $A \in M_{n\times n}(\mathbb{F})$ and $B \in M_{n\times n}(\mathbb{F})$. Define the Frobenius inner product as:

$$\langle A, B \rangle = tr(B^* A)$$

[Recall: $tr(A) = \sum_{i=1}^{n} A_{ii}$ (Sum over diagonal entries)

Remark:
- (a), (b), (c) and (d) are valid for Frobenius norm.
- We will verify (a) and (d)

(a):
$$\begin{aligned}\langle A+B, C \rangle &= tr(C^*(A+B)) \\ &= tr(C^*A + C^*B) \\ &= tr(C^*A) + tr(C^*B) \\ &= \langle A, C\rangle + \langle B, C\rangle.\end{aligned}$$

(d):
$$\begin{aligned}\langle A, A \rangle = tr(A^*A) &= \sum_{i=1}^{n} (A^*A)_{ii} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik} A_{ki} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{ki}}\, A_{ki} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2 \geq 0\end{aligned}$$

If $A \neq 0$, $A_{ki} \neq 0$ for some k and i.
Thus, $\langle A, A\rangle > 0$

Theorem 1: Let V = inner product space. Let $x, y, z \in V$ and $c \in \mathbb{F}$. Then:

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(d) $\langle x, x \rangle = 0$ iff $x = 0$

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all x, then $y = z$.

Proof: We'll prove (a) and (c) only.

(a) $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$

$= \langle x, y \rangle + \langle x, z \rangle$

(c) $\langle x, \underset{\substack{\| \\ \langle x, 0 \rangle}}{0+0} \rangle = \langle x, 0 \rangle + \langle x, 0 \rangle$ $\quad \therefore \langle x, 0 \rangle = 0$.

Definition 2: V = inner product space. Define the norm or length of $x \in V$ by $\|x\| = \sqrt{\langle x, x \rangle}$.

Theorem 2: Let V = inner product space. For $\forall x, y \in V$ and $c \in \mathbb{F}$, we have:

(a) $\|cx\| = |c| \, \|x\|$

(b) $\|x\| = 0 \Leftrightarrow x = 0$ $\quad (\|x\| \geq 0)$

(c) (Cauchy-Schwarz inequality) $|\langle x, y \rangle| \leq \|x\| \, \|y\|$

(d) (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$

Proof: (a) $\sqrt{\langle cx, cx \rangle} = \|cx\| = \sqrt{c \langle x, cx \rangle}$

$= \sqrt{c \bar{c} \langle x, x \rangle} = \sqrt{|c|^2 \|x\|^2}$

$= |c| \, \|x\|$

(b) Part (d) of Theorem 1.

(c) If $y=0$, trivial. Assume $y \neq 0$. Then for $\forall c \in \mathbb{F}$,

$$0 \leq \|x-cy\|^2 = \langle x-cy, x-cy \rangle = \langle x, x-cy \rangle - c\langle y, x-cy \rangle$$
$$= \langle x,x \rangle - \bar{c}\langle x,y \rangle - c\underbrace{\langle y,x \rangle}_{\overline{\langle x,y \rangle}} + c\bar{c}\langle y,y \rangle$$

Let $c = \frac{\langle x,y \rangle}{\langle y,y \rangle}$. We have:

$$0 \leq \langle x,x \rangle - \frac{|\langle x,y \rangle|^2}{\langle y,y \rangle} - \cancel{\frac{|\langle x,y \rangle|^2}{\langle y,y \rangle}} + \cancel{\frac{|\langle x,y \rangle|^2}{\langle y,y \rangle}}$$

$\therefore \quad |\langle x,y \rangle| \leq \|x\| \|y\|$.

(d) $$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$$
$$= \langle x,x \rangle + 2Re(\langle x,y \rangle) + \langle y,y \rangle$$
$$\leq \|x\|^2 + 2|\langle x,y \rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$\therefore \quad \|x+y\| \leq \|x\| + \|y\|$.

<u>Remark</u>: Using the theorem, consider $\mathbb{F}^n$ with $\langle \vec{x}, \vec{y} \rangle = \sum x_i \bar{y}_i$.

Then:
- $$\left| \sum_{i=1}^n a_i \bar{b}_i \right| \leq \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n |b_i|^2 \right)^{\frac{1}{2}}$$
- $$\left[\sum_{i=1}^n |a_i + b_i|^2 \right]^{\frac{1}{2}} \leq \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^n |b_i|^2 \right)^{\frac{1}{2}}$$

<u>Definition 3</u>: V = inner product space. We say $\vec{x} \in V$ and $\vec{y} \in V$ are orthogonal or perpendicular if $\langle \vec{x}, \vec{y} \rangle = 0$. A subset S is orthogonal if any two distinct vectors in S is orthogonal. S is called orthonormal if S is orthogonal and for $\forall \vec{x} \in S$, $\|\vec{x}\| = 1$.

Example 1: Consider $V = \mathbb{F}^{2n}$.

Then: $\{(1,-1,0,\dots,0), (0,0,1,-1,0,\dots,0), \dots, (0,\dots,0,1,-1)\}$ is an orthogonal set. We can normalize it to make it orthonormal:

$$\left\{ \frac{(1,-1,0,\dots,0)}{\sqrt{2}}, \frac{(0,0,1,-1,0,\dots,0)}{\sqrt{2}}, \dots, \frac{(0,\dots,0,1,-1)}{\sqrt{2}} \right\}$$

Example 2: Let $V = \{ f : [0,2\pi] \to \mathbb{C} : f \text{ is continuous} \}$

Let $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx$

Let $f_n(t) = e^{int}$ where $0 \le t \le 2\pi$ and $i = \sqrt{-1}$.

$= \cos nt + i \sin nt$

Then: for $m \neq n$, we have:

$$\langle f_m, f_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt} \overline{e^{int}} \, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t} \, dt = 0$$

Also, $\langle f_m, f_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt} e^{-imt} \, dt = 1$.

Thus, $\{ f_1, f_2, \dots, f_m, \dots \}$ forms an orthonormal set.

Definition 4: Let V be an inner product space. A subset of V is called an orthonormal basis if it is an orthonormal set and it is an ordered basis.

e.g.
- $\left\{ \frac{(1,-1)}{\sqrt{2}}, \frac{(1,1)}{\sqrt{2}} \right\}$ is an orthonormal basis
- Standard ordered basis is an orthonormal basis.

Theorem 3: Let V be an inner product space. Suppose $S=\{\vec{v}_1,\ldots,\vec{v}_k\}$ is an orthogonal subset of $\vec{v}$ where $\vec{v}_i \neq 0$ for all i. If $\vec{y} \in Span(S)$, then:

$$\vec{y} = \sum_{i=1}^{k} \frac{\langle \vec{y}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$$

Proof: Let $y = \sum_{i=1}^{k} a_i \vec{v}_i$ where $a_1, a_2, \ldots, a_k \in \mathbb{F}$.

For all j,

$$\langle \vec{y}, \vec{v}_j \rangle = \langle \sum_{i=1}^{k} a_i \vec{v}_i, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle$$

$$= a_j \|\vec{v}_j\|^2.$$

$$\therefore \quad a_j = \frac{\langle \vec{y}, \vec{v}_j \rangle}{\|\vec{v}_j\|^2}.$$

Theorem 4: An orthogonal subset S (without $\vec{0}$) is linearly independent.

Proof: Let $\vec{0} = \sum_{i=1}^{k} a_i \vec{v}_i$ where $\vec{v}_1, \ldots, \vec{v}_k \in S$.

Then: $\langle \vec{0}, \vec{v}_j \rangle = 0 = \langle \sum_{i=1}^{k} a_i \vec{v}_i, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle$

$$= a_j \langle \vec{v}_j, \vec{v}_j \rangle \quad \text{for } \forall j.$$

$\therefore \quad a_j = 0 \quad$ for all j.

Thus, S is linearly independent.