Lecture 10: Direct sum and Invariant subspaces

Theorem 1: Let $W_1, W_2, \ldots, W_k$ be subspaces.
The following are equivalent:

(a) $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$

(b) $V = \sum_{i=1}^{k} W_i$ and $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k = 0 \Rightarrow \vec{v}_1 = \vec{v}_2 = \ldots = \vec{v}_k = \vec{0}$.

(c) For each $\vec{v} \in V$, $\vec{v}$ can be uniquely written as:

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k \quad (\vec{v}_i \in W_i)$$

(d) If γ_i = ordered basis for W_i, then:
$\gamma_1 \dot\cup \gamma_2 \dot\cup \ldots \dot\cup \gamma_k$ is an ordered basis for V.

(e) For each $i = 1, 2, \ldots, k$, $\exists$ ordered basis γ_i for W_i such that:
$\gamma_1 \dot\cup \gamma_2 \dot\cup \ldots \dot\cup \gamma_k$ is an ordered basis for V.

Proof: (a) $\Rightarrow$ (b): Assume $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$. Clearly, $V = W_1 + W_2 + \ldots + W_k$ by the definition of direct sum.

Let $\underset{\overset{\uparrow}{W_1}}{\vec{v}_1} + \underset{\overset{\uparrow}{W_2}}{\vec{v}_2} + \ldots + \underset{\overset{\uparrow}{W_k}}{\vec{v}_k} = \vec{0}$. For each j, $-\vec{v}_j = \sum_{i \neq j} \vec{v}_i \in \sum_{i \neq j} W_i$

But $-\vec{v}_j \in W_j$. $\therefore$ $-\vec{v}_j \in W_j \cap \sum_{i \neq j} W_i = \{\vec{0}\} \Rightarrow \vec{v}_j = \vec{0}$ for $\forall j$

(b) $\Rightarrow$ (c): Assume (b). Let $\vec{v} \in V$. $\exists \vec{v}_1, \ldots, \vec{v}_k \ni \vec{v}_i \in W_i$ and
$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k$.

Need to prove the representation is unique.
Suppose $\vec{v} = \vec{w}_1 + \vec{w}_2 + \ldots + \vec{w}_k$ with $\vec{w}_i \in W_i$.

Then: $\vec{v}-\vec{v}=\vec{0}=(\underbrace{\vec{v}_1-\vec{w}_1}_{\hat{w}_1})+\dots+(\underbrace{\vec{v}_k-\vec{w}_k}_{\hat{w}_k})$

Thus, $(\vec{v}_1-\vec{w}_1)=(\vec{v}_2-\vec{w}_2)=\dots=(\vec{v}_k-\vec{w}_k)=0$

$\Rightarrow \vec{v}_1=\vec{w}_1, \dots, \vec{v}_k=\vec{w}_k$.

(c) $\Rightarrow$ (d): Assume (c). For each i, let γ_i = ordered basis for W_i.

By (c), $V=\sum_{i=1}^{k} W_i$ and thus $\beta=\gamma_1\cup\gamma_2\cup\dots\cup\gamma_k$ generates V.

Now, we need to prove that β is lin. ind.

Let $\gamma_i=\{\vec{v}_{i_1},\dots,\vec{v}_{i m_i}\}$ for each i.

Consider $\sum_{i=1}^{k}\left(\underbrace{\sum_{j=1}^{m_i} a_{ij}\vec{v}_{ij}}_{\text{"}\vec{w}_i\in W_i}\right)=0$.

Thus, $\underbrace{\vec{w}_1}_{\hat{W}_1}+\underbrace{\vec{w}_2}_{\hat{W}_2}+\dots+\underbrace{\vec{w}_k}_{\hat{W}_k}=\vec{0}$. Since $\vec{0}=\vec{0}+\vec{0}+\dots+\vec{0}=\vec{w}_1+\vec{w}_2+\dots+\vec{w}_k$,

we have: $\vec{w}_1=\vec{w}_2=\dots=\vec{w}_k=\vec{0}$.

$\therefore \vec{w}_i=\sum_{j=1}^{m_i} a_{ij}\vec{v}_{ij}=\vec{0}$ for $\forall i$.

But $\{\vec{v}_{i_1},\dots,\vec{v}_{i m_i}\}$ is lin. ind. So, $a_{ij}=0$ for all i, j.

Thus, $\beta=\gamma_1\cup\gamma_2\cup\dots\cup\gamma_k$ is lin ind and so β is a basis of V.

(d) $\Rightarrow$ (e): Obvious.

(e) $\Rightarrow$ (a): Assume (e). Let γ_i = ordered basis of W_i $\Rightarrow$

$\gamma_1\cup\gamma_2\cup\dots\cup\gamma_k$ = ordered basis for V.

Then: $V=\text{span}(\gamma_1\dot{\cup}\gamma_2\dot{\cup}\dots\dot{\cup}\gamma_k)=\text{span}(\gamma_1)+\text{span}(\gamma_2)+\dots+\text{span}(\gamma_k)$

$= W_1 + W_2 + \dots + W_k$.

Let $\vec{v} \neq \vec{0}$ and $\vec{v} \in W_j \cap \sum_{i \neq j} W_i$. Then: $\vec{v} \in W_j$ and $\vec{v} \in \sum_{i \neq j} W_i$

Hence, $\vec{v}$ can be written as nontrivial combination of γ_j

Also, $\vec{v}$ can be written as nontrivial combination of $\bigcup_{i \neq j} \gamma_i$.

So, $\vec{v}$ can be written as lin. combination of $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ in two different ways. Contradicting that $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ is a basis for V. So, $\vec{v} = \vec{0}$. $\therefore W_j \cap \sum_{i \neq j} W_i = \{\vec{0}\}$.

<u>Invariant subspace</u>

<u>Definition 1</u>: Let $T: V \to V$. Let W = subspace of V. W is called T-invariant subspace of V if $T(W) \subseteq W$. That is, $T(\vec{v}) \in W$ for all $\vec{v} \in W$.

<u>Example 1</u>:

- $W = \{\vec{0}\}$ (Since $T(\vec{0}) = \vec{0} \in W$)
- V (Since $T(V) \subseteq V$ is a subspace of V)
- $R(T)$ (Let $\vec{w} \in R(T)$. Then: $\vec{w} = T(\underset{\in V}{\vec{v}})$. $T(\vec{w}) = T^2(\vec{v}) \in R(T)$)
- $N(T)$ (Let $\vec{v} \in N(T)$. Then $T(\vec{v}) = \vec{0} \in N(T)$)
- E_λ (Let $\vec{v} \in E_\lambda$. Then $T(\vec{v}) = \lambda \vec{v}$

  But $T(\lambda \vec{v}) = \lambda T(\vec{v}) = \lambda^2 \vec{v} = \lambda(\lambda \vec{v})$

  $\therefore \lambda \vec{v} \in E_\lambda \Rightarrow T(\vec{v}) \in E_\lambda$)

Example 2: (Projection) Define $T_1: \mathbb{R}^n \to \mathbb{R}^n$ by:

$$T_1(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{n-1}, 0)$$

Then: $W_i = \{(x_1, x_2, \dots, x_i, 0, 0, \dots, 0) : x_1, x_2, \dots, x_i \in \mathbb{R}\}$ $1 \le i \le n-1$ are T_1-invariant. (Check!)

Let $T_2(x_1, x_2, \dots, x_n) = (f_1(x_1, \dots, x_{n-1}), \dots, f_{n-1}(x_1, \dots, x_{n-1}), 0)$ where $f_i(x_1, \dots, x_{n-1})$ are linear functional depending on $x_1, x_2, \dots, x_{n-1}$.

Then W_{n-1} is T_2-invariant. (Check)

Definition 3: Let $T: V \to V$. The subspace:

$W = \operatorname{span}(\{x, T(x), T^2(x), \dots\})$ is called the <u>T-cyclic</u> subspace of V generated by x.

Remark:
- T-cyclic subspace is T-invariant:

  Let $\vec{v} \in W$. Then $\vec{v} = \lambda_1 T^{i_1}(x) + \lambda_2 T^{i_2}(x) + \dots + \lambda_n T^{i_n}(x)$

  Then: $T(\vec{v}) = \lambda_1 T^{i_1+1}(x) + \lambda_2 T^{i_2+1}(x) + \dots + \lambda_n T^{i_n+1}(x) \in W$

- T-cyclic subspace W is the smallest T-invariant subspace containing x.

  Let W_2 = T-invariant subspace containing x.

  Then: $x \in W_2$, $Tx \in W_2$, $\dots$, $T^i(x) \in W_2$, $\dots$

  But W_2 is a subspace, thus,

  $\operatorname{span}\{x, Tx, \dots\} = W \subseteq W$