Quick revision for the midterm examination

Something you need to remember:

1. Imaging is related to image transformation:

$$\mathcal{O} : \mathcal{I} \to \mathcal{I} \qquad (\mathcal{I} = \text{collection of images})$$

Definition: (Linear image transformation)

$\mathcal{O} : \mathcal{I} \to \mathcal{I}$ is linear $\Longleftrightarrow$ $\mathcal{O}(af + g) = a\,\mathcal{O}(f) + \mathcal{O}(g)$ for $\forall f, g \in \mathcal{I}$; $\forall a \in \mathbb{R}$

Let $g = \mathcal{O}(f)$.

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, \underbrace{h(x, \alpha, y, \beta)}_{\text{PSF}} \quad \text{where}$$

$$h(x, \alpha, y, \beta) = \left[\mathcal{O}(P_{xy})\right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{cl} \begin{array}{ccccc} & & y^{th} & & \\ & & \downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow x^{th} \\ \\ \\ \end{array} \end{array}$$

Matrix representation:

$$
\begin{array}{cc}
 & y \longrightarrow \\
\beta \Big\downarrow & H = \left(\begin{array}{cccc}
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=1 \end{pmatrix}\end{array}\right) \\
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=2 \end{pmatrix}\end{array}\right) \\
\vdots & \vdots & & \vdots \\
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=N \end{pmatrix}\end{array}\right) & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=N \end{pmatrix}\end{array}\right) & \cdots & \left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=N \end{pmatrix}\end{array}\right)
\end{array}\right) \in M_{N^2 \times N^2}
\end{array}
$$

Meaning of

$$
\left(\begin{array}{cc} & x\rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j \end{pmatrix}\end{array}\right) : \begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix} \in M_{N\times N}
$$

$$
h(\underbrace{x}_{\text{col of small block}}, \underbrace{\alpha}_{\text{row of small block}}, \underbrace{y}_{\text{col of block matrix}}, \underbrace{\beta}_{\text{row of block matrix}})
$$

Definition: H is called the <u>transformation matrix</u> of $\mathcal{O}$.

- Shift invariant $\Longleftrightarrow$ $h(x, \alpha, y, \beta) = g(\alpha - x, \beta - y)$

How to check ??

$$g = e^{-|\alpha - x|} \cos(\beta - y)$$

$(x, y) \to (\alpha, \beta)$

If g is periodic in both argument, the matrix representation of the PSF is block-circulant:

e.g.

$$H = \begin{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} \\ \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \end{pmatrix}$$

If g is NOT periodic, the matrix representation is block-Toplitz.

Practice midterm 1, 4

What is block-Toplitz??

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{(1,1)} & \mathbf{A}_{(1,2)} & & \cdots & \mathbf{A}_{(1,n-1)} & \mathbf{A}_{(1,n)} \\ \mathbf{A}_{(2,1)} & \mathbf{A}_{(1,1)} & \mathbf{A}_{(1,2)} & & & \mathbf{A}_{(1,n-1)} \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \mathbf{A}_{(2,1)} & \mathbf{A}_{(1,1)} & \mathbf{A}_{(1,2)} & \\ \vdots & & & \ddots & \ddots & \ddots \\ \mathbf{A}_{(n-1,1)} & & & \mathbf{A}_{(2,1)} & \mathbf{A}_{(1,1)} & \mathbf{A}_{(1,2)} \\ \mathbf{A}_{(n,1)} & \mathbf{A}_{(n-1,1)} & \cdots & & \mathbf{A}_{(2,1)} & \mathbf{A}_{(1,1)} \end{bmatrix}$$

← Each block matrix is Toplitz:

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

↑ Block-Toplitz

e.g.

$$\begin{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \\ \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \end{pmatrix}$$

← Block-Toplitz.

Why??

Consider

$$A_{ij} = \begin{array}{cc} & x \rightarrow \\ \begin{array}{c}\alpha \\ \downarrow\end{array} & \left(\begin{array}{c} \left(\begin{array}{c} y = j \\ \beta = i \end{array} \right) \end{array} \right) \end{array}$$

$$\therefore \; A_{ij} = \begin{pmatrix} h(1,1,j,i) & h(2,1,j,i) & \cdots & h(N,1,j,i) \\ h(1,2,j,i) & h(2,2,j,i) & \cdots & h(N,2,j,i) \\ \vdots & \vdots & & \vdots \\ h(1,N,j,i) & h(2,N,j,i) & \cdots & h(N,N,j,i) \end{pmatrix}$$

Shift-invariant $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g(\alpha - x, \beta - y)$ for some g.

$$\therefore \; A_{ij} = \begin{pmatrix} g(0, i-j) & g(\overset{N-1}{\cancel{-1}}, i-j) & \cdots & g(\overset{1}{\cancel{1-N}}, i-j) \\ g(1, i-j) & g(0, i-j) & \cdots & g(\overset{2}{\cancel{2-N}}, i-j) \\ \vdots & \vdots & & \vdots \\ g(N-1, i-j) & g(N-2, i-j) & \cdots & g(0, i-j) \end{pmatrix} \leftarrow \text{Circulant}$$

(Assume periodic property)

- Shift-invariant $\Longleftrightarrow$ convolution

$$\left[\, f * g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, g(\alpha-x,\beta-y) \,\right]$$

Practice midterm : 2, 3

- Separable $h \Longleftrightarrow h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$

$\therefore\; g = h_c^T s = h_c^T f\, h_r$ (Matrix form)

- if H is separable: $H = \begin{pmatrix} g_2(1,1)G_1 & g_2(2,1)G_1 \\ g_2(1,2)G_1 & g_2(2,2)G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_1(2,2) \end{pmatrix}$

Chapter 1 exercise : 10, 12

Image decomposition:

coefficient matrix

If $f = A\, g\, B^T$ where $A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_N \\ | & | & & | \end{pmatrix}$; $B = \begin{pmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_N \\ | & | & & | \end{pmatrix}$

then:

$$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{a}_i \vec{b}_j^{T}}_{\text{elementary images.}}$$

- SVD: $f = U \Sigma V^T$

diagonal matrix (singular-values)

$$f = \sum_{i=1}^{r} \Sigma_{ii}\, \vec{u}_i \vec{v}_i^{T}$$ where $r = \text{rank}(f)$.

How to compute SVD

Let $A \in M_{m\times n}$ $(m > n)$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \begin{pmatrix} \overset{=\sigma_1}{\sqrt{\lambda_1}} & & & \\ & \overset{=\sigma_2}{\sqrt{\lambda_2}} & & \\ & & \ddots & \\ & & & \overset{=\sigma_n}{\sqrt{\lambda_n}} \\ \hline & & 0 & \end{pmatrix} \in M_{m\times n}$$

Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, ..., $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Error of the approximation by SVD

Theorem: Let $f = \sum_{j=1}^{r} \sigma_j \vec{u}_j \vec{v}_j^T$ be the SVD of a $M \times N$ image f. For any $k < r$,

and $f_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^T$, we have: $\| f - f_k \|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$

λ_i

Practice midterm = 7

$$f_k = U \begin{pmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_k & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} V^T$$

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

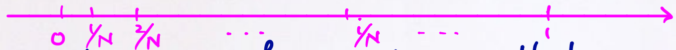
$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots$; $n = 0, 1, 2, \dots, 2^p - 1$

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \dots, N-1$.

We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{H} f \tilde{H}^T.$$

$$\tilde{H}^T \tilde{H} = \tilde{H} \tilde{H}^T = I$$

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T \underset{\uparrow}{g} \tilde{H}$$

$g \leftarrow$ transformed image

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_1^T & - \\ - & \vec{h}_2^T & - \\ & \vdots & \\ - & \vec{h}_N^T & - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{= I_{ij}^H}$

I_{ij}^T = elementary images under Haar Transform.

Practice midterm: 8

Definition: (Walsh function) The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \left\{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \ldots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

The Walsh transform of a $N \times N$ image is defined as follows.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

The Walsh transform matrix is: $\tilde{W} \equiv \frac{1}{\sqrt{N}} W$ where $W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{W} f \tilde{W}^T$$

$$\tilde{W}^T \tilde{W} = I = \tilde{W} \tilde{W}^T$$

Elementary images under Walsh transform:

Under Walsh Transform, $f = \tilde{W}^T g \tilde{W}$.

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{W}_i \vec{W}_j^T}_{"\, I_{ij}^W}$ where $\tilde{W} = \begin{pmatrix} - \vec{W}_1^T - \\ - \vec{W}_2^T - \\ \vdots \\ - \vec{W}_N^T - \end{pmatrix}$

I_{ij}^W = elementary images under Walsh transform.

Practice midterm : 9, 10.

- DFT

- The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \leq k \leq M-1$, $0 \leq l \leq N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j 2\pi \left(\frac{km}{M} + \frac{ln}{N} \right)}$$

$$e^{j\theta} = \cos\theta + j \sin\theta$$

$\sqrt{-1}$

- The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j 2\pi \left(\frac{pm}{M} + \frac{qn}{N} \right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

• Define $U_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \le k, l \le N-1$ and $U = (U_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$

U is clearly symmetric and also :

$$\hat{g} = U g U \quad \text{(DFT in matrix form)}$$

Image decomposition

has special structure → allow FFT.

$$U U^* = U^* U = \frac{1}{N} I, \qquad U^* = \overline{U}^T$$

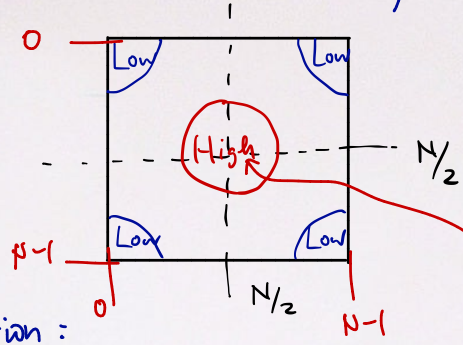
Practice midterm :

11, 12

• DFT of $g * w(p,q) = MN \, DFT(g)(p,q) DFT(w)(p,q)$

Easier to handle convolution in the frequency domain

- Make sure you can understant why :



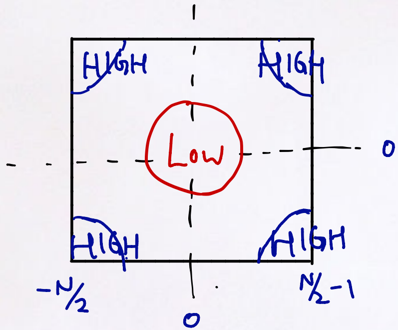
(Lecture 7 material)

$$e^{j\frac{2\pi}{N}\cdot\left[\left(\frac{N}{2}\right)k+\left(\frac{N}{2}\right)l\right]}$$

$$=(-1)^{k+l}$$

highest frequency !!

- After centralization :



Example: Reconstruct an image from DFT(I) using only 3 frequencies closest to 0

$$I = \left(I(m,n)\right)_{0 \le m,n \le 3}$$

After centralization:

$$\begin{array}{cccccccc|c}
a & b & c & d & a & b & c & d & -4 \\
e & f & g & h & e & f & g & h & -3 \\
i & j & k & l & i & j & k & l & -2 \\
m & n & o & p & m & n & o & p & -1 \\
a & b & c & d & a & b & c & d & 0 \\
e & f & g & h & e & f & g & h & 1 \\
i & j & k & l & i & j & k & l & 2 \\
m & n & o & p & m & n & o & p & 3 \\
\hline
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 &
\end{array}$$

Distance of (yellow) from $(0,0) = \sqrt{m^2+n^2} = 0$

Distance of (pink) from $(0,0) = \sqrt{m^2+n^2} = 1$

Distance of (green) from $(0,0) = \sqrt{m^2+n^2} = \sqrt{2}$

$\therefore$ We keep (yellow) (green) (pink)

Do Practice midterm: 11. For further understanding, 12, 13, 14.

Three commonly used filter:

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \leq D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

(Ringing)

2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer):

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0^2\right)^n}$$

(No visible ringing)

3. Gaussian low-pass filter

$$H(u,v) = \exp\left(- \frac{\overset{u^2+v^2}{\overset{\shortparallel}{D(u,v)}}}{2\sigma^2}\right)$$

(No visible ringing)

σ = spread of the Gaussian function

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if} \quad D(u,v) \le D_0^2 \\ 1 & \text{if} \quad D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n}$$

$(H(u,v) = 0 \text{ if } D(u,v) = 0)$

Choose the right n

Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!