mmat5390: mathematical image processing

assignment 2 solutions

1. (a) $\tilde{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$

   (b)

$$\begin{aligned}
A_{\text{Haar}} &= \tilde{H}A\tilde{H}^T \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 4 & 3 \\ 2 & 4 & 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 8 & 9 & 14 & 8 \\ -2 & -3 & -4 & -2 \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} & -3\sqrt{2} \\ \sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{39}{4} & -\frac{5}{4} & -\frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{2} \\ -\frac{11}{4} & \frac{1}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 & 2 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{3}{2} & -1 \end{pmatrix}.
\end{aligned}$$

Then there are 4 such modified Haar transform. We choose one of them for explanation.

A'_{Haar} is $\begin{pmatrix} \frac{39}{4} & -\frac{5}{4} & 0 & \frac{3\sqrt{2}}{2} \\ -\frac{11}{4} & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 & 2 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{3}{2} & -1 \end{pmatrix}$, and thus:

$$\begin{aligned}
\tilde{A} &= \tilde{H}^T A'_{\text{Haar}}\tilde{H} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{39}{4} & -\frac{5}{4} & 0 & \frac{3\sqrt{2}}{2} \\ -\frac{11}{4} & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 & 2 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 6 & -\frac{1}{4} & -\frac{3\sqrt{2}}{4} & 3\sqrt{2} \\ 8 & -\frac{9}{4} & \frac{5\sqrt{2}}{4} & -\sqrt{2} \\ 12 & -\frac{7}{4} & \frac{5\sqrt{2}}{4} & \sqrt{2} \\ 13 & -\frac{3}{4} & -\frac{7\sqrt{2}}{4} & 3\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{17}{16} & \frac{29}{16} & \frac{49}{16} & \frac{1}{16} \\ \frac{33}{16} & \frac{13}{16} & \frac{33}{16} & \frac{49}{16} \\ \frac{51}{16} & \frac{31}{16} & \frac{63}{16} & \frac{47}{16} \\ \frac{35}{16} & \frac{63}{16} & \frac{79}{16} & \frac{31}{16} \end{pmatrix}.
\end{aligned}$$

   (c)

```
%% Q2D
function H = q2d(n)
N = 2^n;
H = zeros(N);
H(1,:) = 1; % H_0(t) = 1
for k = 1:N-1
```

```
        p = floor(log2(k));
        H_k = Haar(p, k-2^p);
        for i = 0:N-1
            t = i/N;
            if H_k(1,1) <= t && t < H_k(1,2)
                H(k+1, i+1) = H_k(1,3);
            elseif H_k(2,1) <= t && t < H_k(2,2)
                H(k+1, i+1) = H_k(2,3);
            end
        end
    end
    H = H/sym(sqrt(N));
    end

    %% HAAR
    % Create Haar function H_{2^p+n}(t)
    % The output $H_{2^p+n}(t)$ is represented by a matrix:
    % [n/(2^p), (n+0.5)/(2^p), sqrt(2)^p; (n+0.5)/(2^p), (n+1)/(2^p), -sqrt(2)^p]
    function H = Haar(p, n)
    H = [n/(2^p) (n+1/2)/(2^p) sqrt(2)^p; (n+0.5)/(2^p) (n+1)/(2^p) -sqrt(2)^p];
    end
```

q2d.m

(d)

```
    %% Q2E
    function A2 = q2e(A,p)
    N = size(A,1);
    n = round(log2(N));
    H = q2d(n);
    num_ps_to_keep = floor(p/100*N^2);
    A_Haar = H*A*H';
    pos = findkPos(A_Haar, N^2-num_ps_to_keep);
    for p = pos
        A_Haar(p(1),p(2)) = 0;
    end
    A2 = H'*A_Haar*H;
    end

    %% FINDKPOS
    % Find the positions of the entries with the k smallest absolute values in a matrix A
    function pos = findkPos(A, k)
    pos = zeros(2,k);
    [m, ~] = size(A);
    B = abs(A(:));
    for i = 1:k
        ind = find(B == min(B));
        ind = ind(1);
        B(ind) = Inf;

        c = floor((ind-1)/m)+1; r = ind-(c-1)*m;
        pos(:,i) = [r;c];
    end
    end
```

q2e.m

2. (a) $\tilde{W} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

(b)

$$
\begin{aligned}
B_{\text{Walsh}} &= \tilde{W} B \tilde{W}^T \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 5 & 4 \\ 2 & 2 & 3 & 2 \\ 5 & 6 & 1 & 3 \\ 2 & 3 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 10 & 12 & 13 & 11 \\ 4 & 6 & -3 & -1 \\ 4 & 4 & -5 & -1 \\ 2 & 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 46 & 2 & 4 & 0 \\ 6 & -14 & 0 & -4 \\ 2 & -14 & -4 & -4 \\ 6 & -2 & -4 & -4 \end{pmatrix}.
\end{aligned}
$$

Then there are 4 such modified Haar transform. We choose one of them for explanation. B'_{Walsh} is $\frac{1}{4}\begin{pmatrix} 46 & 0 & 0 & 0 \\ 6 & -14 & 0 & -4 \\ 0 & -14 & -4 & -4 \\ 6 & 0 & -4 & -4 \end{pmatrix}$, and thus:

$$
\begin{aligned}
\tilde{B} &= \tilde{W}^T B'_{\text{Walsh}} \tilde{W} \\
&= \frac{1}{16}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 46 & 0 & 0 & 0 \\ 6 & -14 & 0 & -4 \\ 0 & -14 & -4 & -4 \\ 6 & 0 & -4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{32}\begin{pmatrix} 11 & 7 & 39 & 35 \\ 19 & 15 & 19 & 15 \\ 41 & 45 & 5 & 25 \\ 21 & 25 & 29 & 17 \end{pmatrix}.
\end{aligned}
$$

3. (a) $\int_{\mathbb{R}}[H_0(t)]^2\,dt = \int_0^1 dt = 1$.
For any $p \in \mathbb{N}\setminus\{0\}$ and $n \in \mathbb{Z}\cap[0, 2^p-1]$,

$$
\begin{aligned}
\int_{\mathbb{R}}[H_{2^p+n}(t)]^2\,dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} (2^{\frac{p}{2}})^2\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})^2\,dt \\
&= 2\cdot\frac{1}{2^{p+1}}\cdot 2^p = 1.
\end{aligned}
$$

(b) i. Let $m \in \mathbb{N}\setminus\{0\}$. There exists $p \in \mathbb{N}\cup\{0\}$ and $n \in \mathbb{Z}\cap[0,2^p-1]$ such that $m = 2^p+n$. Then

$$
\begin{aligned}
\langle H_0, H_m\rangle &= \int_{\mathbb{R}} H_0(t)H_{2^p+n}(t)\,dt \\
&= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})\,dt \\
&= \frac{1}{2^{p+1}}\cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}}\cdot(-2^{\frac{p}{2}}) = 0.
\end{aligned}
$$

ii. A. Suppose $p_1 = p_2$. Then

$$
\begin{aligned}
\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t)H_{2^{p_1}+n_2}(t)\,dt \\
&= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}}\cdot 0\,dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n+1}{2^{p_1}}} (-2^{\frac{p_1}{2}})\cdot 0\,dt \\
&+ \int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0\cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0\cdot(-2^{\frac{p_1}{2}})\,dt = 0.
\end{aligned}
$$

B. Suppose $p_1 < p_2$. Then either

- $2^{p_2-p_1}n_1 \le n_2 < 2^{p_2-p_1}(n_1+0.5)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$;

  or
- $2^{p_2-p_1}(n_1+0.5) \le n_2 < 2^{p_2-p_1}(n_1+1)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$;

  or
- $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right) = \varnothing$.

In any case, H_{m_1} is constant on $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$, and thus denoting the constant by c,

$$\begin{aligned}\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_2}+n_2}(t)\,dt \\ &= c\int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\,dt + c\int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} (-2^{\frac{p_2}{2}})\,dt \\ &= c[\frac{1}{2^{p_2+1}}\cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}}\cdot(-2^{\frac{p_2}{2}})] = 0.\end{aligned}$$

4. (a) $U_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $U_4 = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}$.

   (b)

$$\begin{aligned}\hat{I} &= U_2 I U_2 \\ &= \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3 & 9\\ 4 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix}7 & 12\\ -1 & 6\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix}19 & -5\\ 5 & 7\end{pmatrix}\end{aligned}$$

   (c) i.

$$\begin{aligned}\hat{J} &= U_4 J U_4 \\ &= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix}4 & -2-2j & 0 & -2+2j\\ -2-2j & 2j & 0 & 2\\ 0 & 0 & 0 & 0\\ -2+2j & 2 & 0 & -2j\end{pmatrix}\end{aligned}$$

   ii.

$$\hat{J}' = \frac{1}{8}\begin{pmatrix}2 & -1-j & 0 & -1+j\\ -1-j & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1+j & 0 & 0 & 0\end{pmatrix}$$

Thus,

$$
\begin{aligned}
J' &= \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 2 & -1-j & 0 & -1+j \\ -1-j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1+j & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}
\end{aligned}
$$