MMAT5390: Mathematical Image Processing

Assignment 2

Due: 8 March 2021

Please give reasons in your solutions.

1. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.
   (a) Write down the Haar transform matrix $\tilde{H}$ for 4×4 images.
   (b) Suppose $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 4 & 3 \\ 2 & 4 & 5 & 2 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the two smallest (in absolute value) nonzero entries of A_{Haar} to 0.
   (c) (Programming, optional) Write a MATLAB function such that
   $$H = q2d(n);$$
   returns the $2^n \times 2^n$ Haar transform matrix.
   (d) (Programming, optional) Write a MATLAB function such that given $A \in M_{2^n \times 2^n}(\mathbb{R})$ and $p \in [0, 100]$,
   $$A2 = q2e(A, p);$$
   returns the image reconstructed from the largest (in absolute value) $p\%$ (rounded to the nearest integer) of the entries of the Haar transform of A. You may make use of the function q2d.

2. Let $W_n(t)$ be the n^{th} Walsh function, where $n \in \mathbb{N} \cup \{0\}$.
   (a) Write down the Walsh transform matrix $\tilde{W}$ for 4×4 images.
   (b) Suppose $B = \begin{pmatrix} 1 & 1 & 5 & 4 \\ 2 & 2 & 3 & 2 \\ 5 & 6 & 1 & 3 \\ 2 & 3 & 4 & 2 \end{pmatrix}$. Compute the Walsh transform B_{Walsh} of B, and compute the reconstructed image $\tilde{B}$ after setting the four smallest (in absolute value) nonzero entries of B_{Walsh} to 0.

3. Let $\mathcal{H} := \{H_m : m \in \mathbb{N} \cup \{0\}\}$ be the sequence of Haar functions.
   Consider the inner product space $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ where
   $$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}} f^2 < \infty \right\},$$
   and for any $f, g \in L^2(\mathbb{R})$,
   $$\langle f, g \rangle = \int_{\mathbb{R}} fg.$$
   (a) (Unit) Prove that $\int_{\mathbb{R}} [H_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$.
   (Hence $H_m \in L^2(\mathbb{R})$ and $\|H_m\| = 1$.)
   (b) (Orthogonality)
      i. Prove that $\langle H_0, H_m \rangle = 0$ for any $m \in \mathbb{N} \setminus \{0\}$.

ii. Let $m_1, m_2 \in \mathbb{N}$ such that $0 \neq m_1 < m_2$. Then $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$ for some $p_1, p_2 \in \mathbb{N} \cup \{0\}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.

A. Suppose $p_1 = p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
Hint. In this case $n_1 < n_2$.

B. Suppose $p_1 < p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
Hint. Consider the possible subset relations between the supports of H_{m_1} and H_{m_2}.

The above establishes that $\mathcal{H}$ is orthonormal in $(L^2(\mathbb{R}), \langle \cdot, \cdot, \rangle)$.

4. (a) Write down the matrices U_2 and U_4 used to calculate the DFTs of 2×2 and 4×4 images respectively, i.e. for any $f \in M_{2\times 2}$ and $g \in M_{4\times 4}$, $\hat{f} = U_2 f U_2$ and $\hat{g} = U_4 g U_4$.

(b) Suppose $I = \begin{pmatrix} 3 & 9 \\ 4 & 3 \end{pmatrix}$. Compute $\hat{I} = DFT(I)$.

(c) Suppose $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

i. Compute $\hat{J} = DFT(J)$.

ii. Compute the real part of $J' = iDFT(\hat{J}')$, where

$$\hat{J}' = \begin{pmatrix} \hat{J}(0,0) & \hat{J}(0,1) & 0 & \hat{J}(0,3) \\ \hat{J}(1,0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{J}(3,0) & 0 & 0 & 0 \end{pmatrix}.$$

Remark. $\mathrm{Re}(J')$ is obtained from J by applying an ideal low-pass filter with radius $D_0 \in [1, \sqrt{2})$.