Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \leq n$.

We need the following theorem.

Theorem: Let $B \in M_{n \times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} — & \vec{v}_1^T & — \\ — & \vec{v}_2^T & — \\ & \vdots & \\ — & \vec{v}_n^T & — \end{pmatrix}$$

Note that $gg^T \in M_{m \times m}$ and $g^T g \in M_{n \times n}$ are symmetric.

$\therefore \exists$ n pairwise orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of $g^T g$.

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i (g\vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i \vec{v}_i) = \lambda_i$.

$$\therefore \|g\vec{v}_i\| = \sigma_i$$

Define $\vec{u}_i = \dfrac{g\vec{v}_i}{\sigma_i}$. Then: $\|\vec{u}_i\| = 1$.

Also,
$$\vec{u}_i \cdot \vec{u}_j = \frac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i\sigma_j} = \frac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i\sigma_j} = \frac{\lambda_j \overbrace{\vec{v}_i^T \vec{v}_j}^{=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}}}{\sigma_i\sigma_j} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases}\sigma_i & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$

In matrix form,
$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r\times m} \underbrace{g}_{m\times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n\times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}$$

Extend $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$.

Extend $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$.

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m\times m} \underbrace{g}_{m\times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n\times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}}_{m\times n} \neq \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^T U = U U^T = I$; $V^T V = V V^T = I$. $\therefore$ $g = U \Lambda^{1/2} V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Recap on the proof of existence:

<u>**Theorem: (Existence of SVD)**</u> Every $m \times n$ image has a SVD.

<u>**Proof:**</u> Consider the case when $m \le n$.

We need the following theorem.

<u>Theorem:</u> Let $B \in M_{n\times n}$ be a real symmetric matrix. Then, $\exists$ orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

Note that $gg^T \in M_{m\times m}$ and $g^Tg \in M_{n\times n}$ are symmetric.

$\therefore \exists$ n pairwise $\boxed{\text{orthonormal eigenvectors } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \text{ of } g^Tg.}$

Find orthonormal basis of g^Tg

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i\vec{v}_i) = \lambda_i(g\vec{v}_i)$.

$\therefore$ $g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i.

Note that g^Tg is positive-definite and hence all of its eigenvalues must be +ve.

$\therefore$ $\lambda_i > 0$ for $i = 1, 2, \dots, r$.

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i\vec{v}_i) = \lambda_i$.

Define $\vec{u}_i$

$$\therefore \|g\vec{v}_i\| = \sigma_i$$

$\boxed{\text{Define } \vec{u}_i = \dfrac{g\vec{v}_i}{\sigma_i}}$. Then: $\|\vec{u}_i\| = 1$.

$$\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Also, $$\vec{u}_i \cdot \vec{u}_j = \frac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i\sigma_j} = \frac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i\sigma_j} = \frac{\lambda_j \vec{v}_i^T\vec{v}_j}{\sigma_i\sigma_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T\vec{u}_j = \begin{cases} \sigma_i & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

In matrix form,

$$\boxed{\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \end{pmatrix}}_{r\times m} \underbrace{g}_{m\times n} \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_r \\ | & | & & | \end{pmatrix}}_{n\times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}}$$

Form preliminary matrix decomposition

$$\boxed{\begin{array}{l}\text{Extend } \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\} \text{ to an orthonormal basis } \{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\} \text{ of } \mathbb{R}^m. \\ \text{Extend } \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\} \text{ to an orthonormal basis } \{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\} \text{ of } \mathbb{R}^n.\end{array}}$$

Extend basis

Then:

$$\underbrace{\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_r^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix}}_{m \times m} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \cdots & \vec{v}_n \\ | & & | & & | \end{pmatrix}}_{n \times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}}_{m \times n} = \Lambda^{1/2}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^T U = U U^T = I$; $V^T V = V V^T = I$. $\therefore$ $g = U \Lambda^{1/2} V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

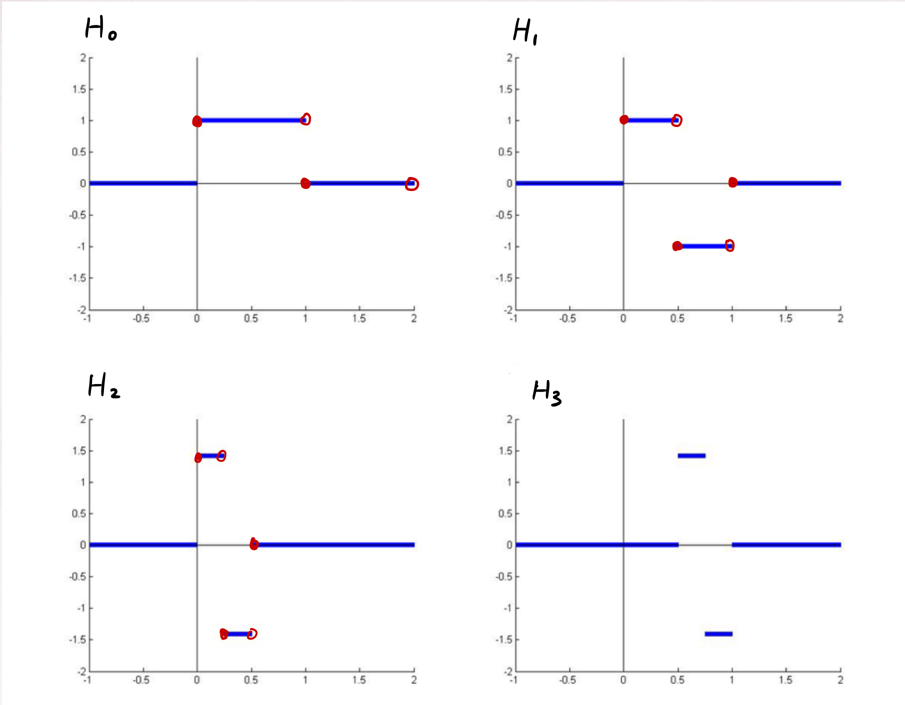
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \dots ; \; n = 0, 1, 2, \dots, 2^p - 1$

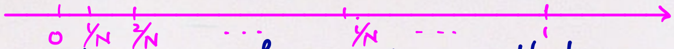
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

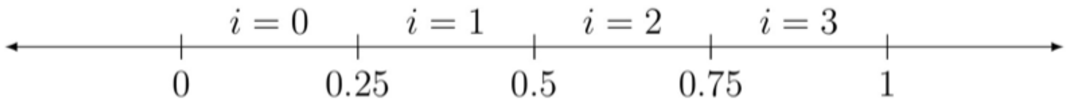
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

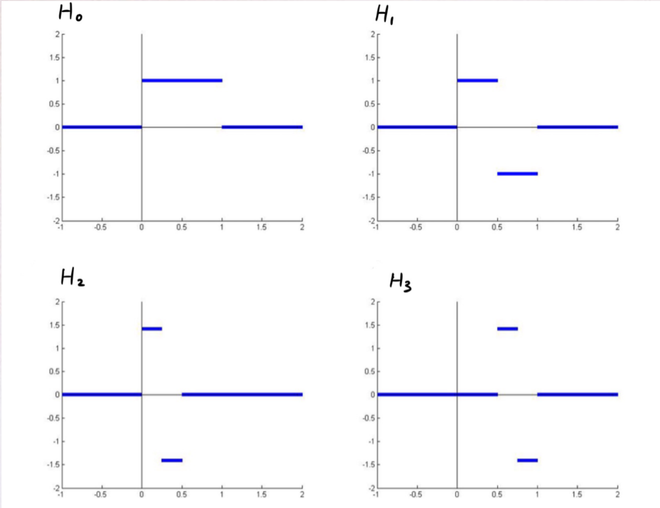
$$g = \tilde{H} f \tilde{H}^T.$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Need to check:



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

More zeros

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T f \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \, \tilde{H}$$

g ← transformed image

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_1^T & - \\ - & \vec{h}_2^T & - \\ & \vdots & \\ - & \vec{h}_N^T & - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{= I^H_{ij}}$

I^T_{ij} = elementary images under Haar Transform.

Definition: (Walsh function) The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \left\{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \ldots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

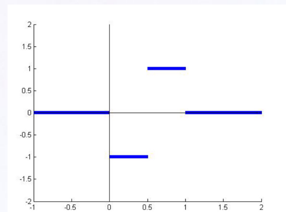
Example: Compute $W_1(t)$.

Put $j = 0$, $q = 1$. Then:

$$W_1(t) = (-1)^{\lfloor 0 \rfloor + 1} \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\} = (-1) \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\}$$

For $0 \le t < \frac{1}{2}$, $W_0(2t) = 1$, $W_0(2t-1) = 0 \Rightarrow W_1(t) = -1$.

For $\frac{1}{2} \le t < 1$, $W_0(2t) = 0$, $W_0(2t-1) = 1 \Rightarrow W_1(t) = 1$.



Definition: (Discrete Walsh transform)

The Walsh Transform of a $N \times N$ image is defined as follows.

Define $W(k, i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

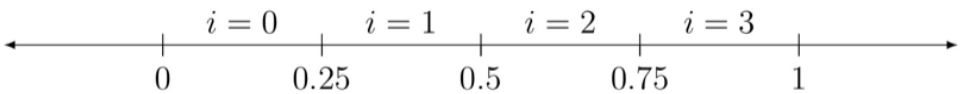
The Walsh transform matrix is: $\tilde{W} \equiv \frac{1}{\sqrt{N}} W$ where $W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

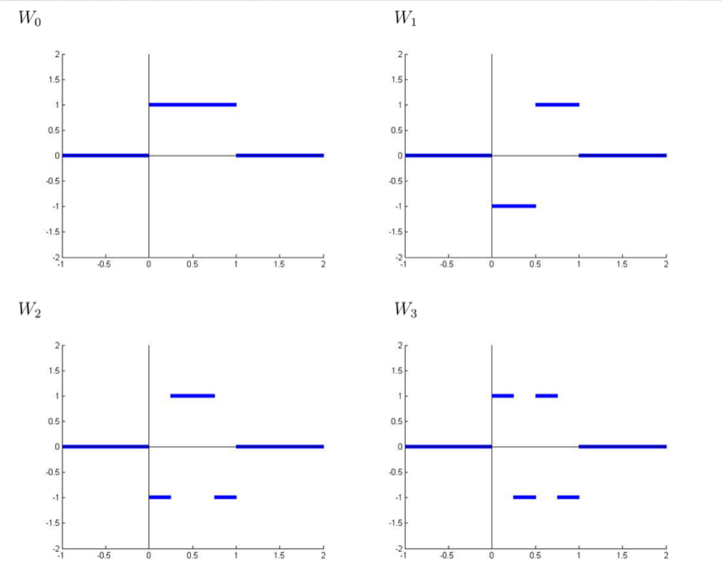
$$g = \tilde{W} f \tilde{W}^T$$

Example Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:



We can check that:



So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}} W = \frac{1}{2} W$$

$(\tilde{W}^T \tilde{W} = I)$

Example 2.7: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{W} f \tilde{W}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

More zeros in the coefficient matrix!

Remark: 1. Walsh transform is to transform an image to a "transformed image" with much more zeros.

Elementary images under Walsh transform:

Under Walsh Transform, $f = \tilde{W}^T g \tilde{W}$.

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{W}_i \vec{W}_j^T$ where $\tilde{W} = \begin{pmatrix} - & \vec{W}_1^T & - \\ - & \vec{W}_2^T & - \\ & \vdots & \\ - & \vec{W}_N^T & - \end{pmatrix}$

$= I_{ij}^W$

I_{ij}^W = elementary images under Walsh transform.

Walsh functions and sine function

Definition: (Rademacher function)

A Rademacher function of order n $(n \neq 0)$ is defined as:

$$R_n(t) \equiv \text{sign}[\sin(2^n \pi t)] \quad \text{for} \quad 0 \le t \le 1.$$

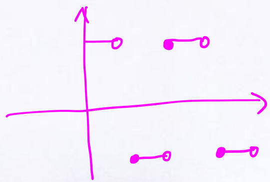
where $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$ and $\text{sign}(x) = 0$ if $x = 0$.

For $n = 0$, $R_0(t) \equiv 1$ for $0 \le x \le 1$.

Let $N = b_{m+1} 2^m + b_m 2^{m-1} + \dots + b_1 2^0$. Then, the R-Walsh function $\tilde{W}_N$ is given by:

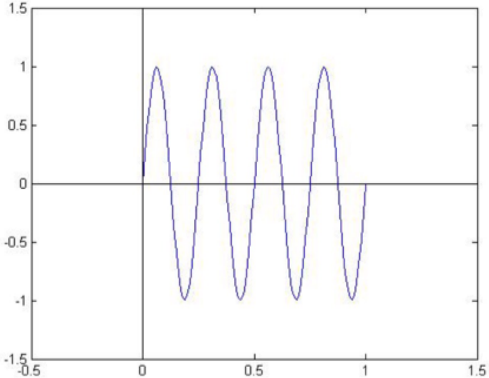
$$\tilde{W}_N = \prod_{i=1,\, b_i \neq 0}^{m+1} R_i(t)$$

(where the values at the jumps are defined such that the function is continuous from the right)

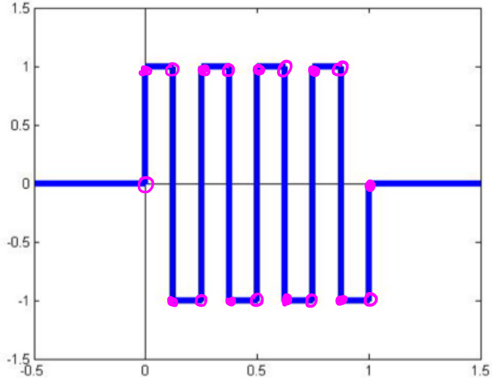


Example : Compute R-Walsh function $\tilde{W}_4$ using Rademacher function.

Consider $\sin(8\pi t)$:



Therefore, $R_3(t) =$



As $4 = \underbrace{1}_{b_3} \cdot 2^2 + \underbrace{0}_{b_2} \cdot 2^1 + \underbrace{0}_{b_1} \cdot 2^0$, we have

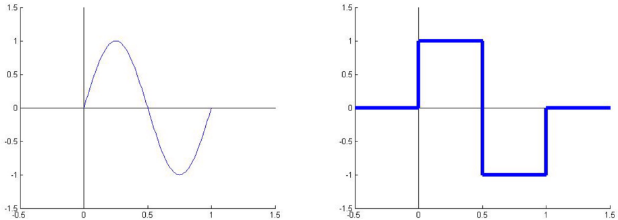
$$\tilde{W}_4 = \prod_{i=1, b_i \neq 0}^{3} R_i(t) = R_3(t)$$

$$(W_{2j+q}(t) \equiv (-1)^{\lfloor j/2 \rfloor + q}\{W_j(2t) + (-1)^{j+q} W_j(2t-1)\})$$

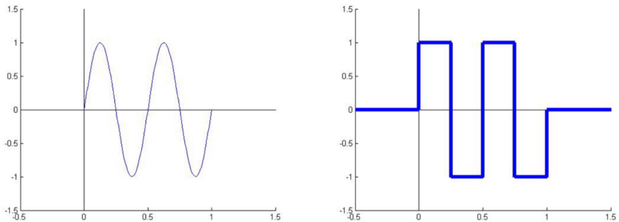
For $W_3(t)$: As $3 = \underbrace{1}_{b_2} \cdot 2^1 + \underbrace{1}_{b_1} \cdot 2^0$, we have

$$\tilde{W}_3(t) = \prod_{i=1, b_i \neq 0}^{2} R_i(t) = R_1(t) R_2(t)$$

$R_1(t)$:



$R_2(t)$:



Therefore, $\tilde{W}_3(t)$: