MMAT5390: Mathematical Image Processing

Solutions to Chapter 3 Exercises

1. (a) As $A = U\Sigma V^T$,
$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$$
and
$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T.$$
Note that
$$\Sigma^T\Sigma = \begin{cases} \begin{pmatrix} \begin{matrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{matrix} & \mathbf{0}_{M\times(N-M)} \\ \mathbf{0}_{(N-M)\times M} & \mathbf{0}_{(N-M)\times(N-M)} \end{pmatrix} & \text{if } M < N \\ \begin{pmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{pmatrix} & \text{if } M \geq N \end{cases}$$
and
$$\Sigma\Sigma^T = \begin{cases} \begin{pmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{pmatrix} & \text{if } M \leq N \\ \begin{pmatrix} \begin{matrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{KK}^2 \end{matrix} & \mathbf{0}_{N\times(M-N)} \\ \mathbf{0}_{(M-N)\times N} & \mathbf{0}_{(M-N)\times(M-N)} \end{pmatrix} & \text{if } M > N \end{cases}$$
Hence $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$ are the square roots of the largest K eigenvalues of $A^T A$ (or AA^T) in descending order, and thus the K-tuple is uniquely determined.

   (b) Suppose $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ are distinct and nonzero. Then each eigenspace of $A^T A$ and AA^T corresponding to eigenvalue σ_{ii}^2 has dimension 1, which means that there are exactly two unit eigenvectors to be chosen from each eigenspace, each being the negative of the other. Such eigenvectors are precisely the first K columns of U and V. Combined with the fact that σ_{ii} are in descending order, the first K columns of U and V are uniquely determined up to a change of sign.

   (c) Suppose $M = N$. Then the argument from (b) follows.

   (d) A counterexample with nondistinct $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ is given by:
$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 I_2 I_2 = U I_2 U^T,$$
where I_2 and $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ are unitary.
A counterexample with $\sigma_{KK} = 0$ is given by:
$$(0\ \ 0) = (1)(0\ \ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(0\ \ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. Let $g \in M_{m\times n}(\mathbb{R})$ with SVD $g = U\Sigma V^T$.

For any $c_1, c_2, \cdots, c_r \in \mathbb{R}$,

$$\sum_{i=1}^{r} c_i\vec{u}_i = \sum_{i=1}^{r} \frac{c_i}{\sigma_i}\sigma_i\vec{u}_i(\vec{v}_i^T\vec{v}_i)$$
$$= (\sum_{i=1}^{r}\sigma_i\vec{u}_i\vec{v}_i^T)(\sum_{j=1}^{r}\frac{c_j}{\sigma_j}\vec{v}_j) \text{ since } \vec{v}_i^T\vec{v}_j = \delta(i-j) \qquad = g(\sum_{j=1}^{r}\frac{c_j}{\sigma_j}\vec{v}_j).$$

Let $\vec{v} \in \mathbb{R}^n$. Since V is unitary, $span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = \mathbb{R}^n$ and thus there exist $c_1, c_2, \cdots, c_n \in \mathbb{R}$ such that $\vec{v} = \sum_{j=1}^{n} c_j\vec{v}_j$. Then

$$\begin{aligned} g\vec{v} &= (\sum_{i=1}^{r}\sigma_i\vec{u}_i\vec{v}_i^T)\sum_{j=1}^{n}c_j\vec{v}_j \\ &= \sum_{i=1}^{r}\sum_{j=1}^{n}c_i\sigma_j\vec{u}_i\vec{v}_i^T\vec{v}_j \\ &= \sum_{i=1}^{r}\sum_{j=1}^{n}c_i\sigma_j\delta(i-j)\vec{u}_i \\ &= \sum_{i=1}^{r}(c_i\sigma_i)\vec{u}_i. \end{aligned}$$

Hence $range(g) = span(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r)$.

For any $c_{r+1}, c_{r+2}, \cdots, c_n \in \mathbb{R}$,

$$\begin{aligned} g(\sum_{i=r+1}^{N}c_i\vec{v}_i) &= (\sum_{j=1}^{r}\sigma_j\vec{u}_j\vec{v}_j^T)(\sum_{i=r+1}^{N}c_i\vec{v}_i) \\ &= \sum_{j=1}^{r}\sum_{i=r+1}^{N}c_i\sigma_j\vec{u}_j\vec{v}_j^T\vec{v}_i \\ &= \sum_{j=1}^{r}\sum_{i=r+1}^{N}c_i\sigma_j\delta(i-j)\vec{u}_j = 0. \end{aligned}$$

Let $\vec{v} = \sum_{i=1}^{n} c_i\vec{v}_i \in \mathbb{R}^n$ such that $g\vec{v} = \vec{0}_m$. Then for any $i \in [1, r]$,

$$\begin{aligned} c_i &= \vec{v}_i^T\vec{v} \\ &= \vec{u}_i^T\vec{u}_i\vec{v}_i^T\vec{v} \\ &= \frac{1}{\sigma_i}\vec{u}_i^T(\sum_{j=1}^{r}\sigma_j\vec{u}_j\vec{v}_j^T)\vec{v} \\ &= \frac{1}{\sigma_i}\vec{u}_i^Tg\vec{v} = \frac{1}{\sigma_i}\vec{u}_i^T\vec{0}_m = 0. \end{aligned}$$

and thus $\vec{v} = \sum_{i=r+1}^{n} c_i\vec{v}_i$.

Hence $null(g) = span\{\vec{v}_{r+1}, \vec{v}_{r+2}, \cdots \vec{v}_n\}$.

3. (a) $ff^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$.

For $\lambda = 20$:

$$\left[\begin{array}{cc|c} -10 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_1 = (0,1)^T$.

For $\lambda = 10$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which gives unit eigenvector $\vec{u}_2 = (1,0)^T$.

Then $\vec{v}_1 = \frac{f^T\vec{u}_1}{\sigma_1} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(0,2,0,1)^T$, and

$\vec{v}_2 = \frac{f^T\vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(1,0,3,0)^T$.

$f^T f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 16 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 4 \end{pmatrix}$.

For $A^T A\vec{v} = 0$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which gives orthonormal eigenvectors $\vec{v}_3 = \frac{1}{\sqrt{10}}(-3,0,1,0)^T$ and $\vec{v}_4 = \frac{1}{\sqrt{5}}(0,1,0,-2)^T$.

Hence an SVD of A is $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & -3 & 0 \\ 2\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 3 & 1 & 0 \\ \sqrt{2} & 0 & 0 & -2\sqrt{2} \end{pmatrix}.$$

(b) The eigenimages are given by

$\vec{u}_1\vec{v}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix}$ and

$\vec{u}_2\vec{v}_2^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence $A = 2\sqrt{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} + \sqrt{10} \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

4. (a) $f^T f = ff^T = \begin{pmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{pmatrix}$.

Denote $a^2+b^2+c^2$ by p, and $ab+bc+ca$ by q.

$$\begin{aligned} \det(f^T f - \lambda I) &= \begin{vmatrix} p-\lambda & q & q \\ q & p-\lambda & q \\ q & q & p-\lambda \end{vmatrix} \\ &= \begin{vmatrix} p-\lambda & q & q \\ q & p-\lambda & q \\ 0 & \lambda+q-p & p-q-\lambda \end{vmatrix} \\ &= (p-q-\lambda) \begin{vmatrix} p-\lambda & q & q \\ q & p-\lambda & q \\ 0 & -1 & 1 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (p-q-\lambda)\begin{vmatrix} p-\lambda & 2q & q \\ q & p+q-\lambda & q \\ 0 & 0 & 1 \end{vmatrix} \\
&= (p-q-\lambda)\begin{vmatrix} p-\lambda & 2q \\ q & p+q-\lambda \end{vmatrix} \\
&= (p-q-\lambda)[\lambda^2-(2p+q)\lambda+p^2+pq-2q^2] \\
&= (p-q-\lambda)(p+\frac{q}{2}+\frac{3|q|}{2})(p+\frac{q}{2}-\frac{3|q|}{2}) \\
&= (p-q-\lambda)^2(p+2q-\lambda) \\
&= (a^2+b^2+c^2-ab-bc-ca-\lambda)^2(a^2+b^2+c^2+2ab+2bc+2ca-\lambda).
\end{aligned}
$$

Note that $p-q=(a-b)^2+(b-c)^2+(c-a)^2$ and $p+2q=(a+b+c)^2$.

Suppose $q \neq 0$.
If $p-q \neq 0$, i.e. a,b,c not all equal,
for $\lambda = p-q = a^2+b^2+c^2-ab-bc-ca$:

$$
\left[\begin{array}{ccc|c} q & q & q & 0 \\ q & q & q & 0 \\ q & q & q & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],
$$

which gives eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{2}}(1,0,-1)^T$ and $\tilde{\vec{v}}_2 = (0,1,-1)^T$;
by Gram-Schmidt orthonormalization, $\vec{v}_2$ for $q \neq 0$ is given by

$$
\begin{cases} \tilde{\vec{v}}_2' = \tilde{\vec{v}}_2 - \langle \vec{v}_1, \tilde{\vec{v}}_2 \rangle \vec{v}_1 = (0,1,-1)^T - \frac{1}{2}(1,0,-1)^T = \frac{1}{2}(-1,2,-1)^T, \\ \vec{v}_2 = \frac{\tilde{\vec{v}}_2'}{\|\tilde{\vec{v}}_2'\|} = \frac{1}{\sqrt{6}}(-1,2,-1)^T. \end{cases}
$$

Then

$$
\begin{aligned}
\vec{u}_1 = \frac{f\vec{v}_1}{\sqrt{p-q}} &= \frac{1}{\sqrt{2(p-q)}}\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(p-q)}}(a-c, c-b, b-a)^T,
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{u}_2 = \frac{f\vec{v}_2}{\sqrt{p-q}} &= \frac{1}{\sqrt{6(p-q)}}\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{6(p-q)}}(2b-c-a, 2a-b-c, 2c-a-b)^T.
\end{aligned}
$$

If $p+2q \neq 0$, i.e. $a+b+c \neq 0$, for $\lambda = p+2q = a^2+b^2+c^2+2ab+2bc+2ca$,

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} -2q & q & q & 0 \\ q & -2q & q & 0 \\ q & q & -2q & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],
\end{aligned}
$$

which give unit eigenvector $\vec{v}_3 = \frac{1}{\sqrt{3}}(1,1,1)^T$.
Then

$$
\begin{aligned}
\vec{u}_3 = \frac{f\vec{v}_3}{\sqrt{p+2q}} &= \frac{1}{\sqrt{3(p+2q)}}\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \frac{a+b+c}{\sqrt{3(p+2q)}}(1,1,1)^T = \frac{1}{\sqrt{3}}(1,1,1)^T.
\end{aligned}
$$

Hence if $q \neq 0$ and $p-q \neq 0$, an SVD of f is given by $f = U\Sigma V^T$, where

$$U = \begin{pmatrix} \frac{a-c}{\sqrt{2(p-q)}} & \frac{2b-c-a}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \\ \frac{c-b}{\sqrt{2(p-q)}} & \frac{2a-b-c}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \\ \frac{b-a}{\sqrt{2(p-q)}} & \frac{2c-a-b}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{p-q} & 0 & 0 \\ 0 & \sqrt{p-q} & 0 \\ 0 & 0 & \sqrt{p+2q} \end{pmatrix}$$

$$\text{and } V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

If $q \neq 0$ and $p-q=0$, then $p+2q \neq 0$. Then since $\vec{v}_1, \vec{v}_2 \in \text{span}\big((1,1,1)^T\big)$, we can use them to form U as well, i.e. $\vec{u}_1 = \vec{v}_1$, $\vec{u}_2 = \vec{v}_2$ and thus $f = V\Sigma V^T$, where V and Σ are the same as above with $\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{p+2q} \end{pmatrix}$.

On the other hand, if $q=0$, $\Sigma = \sqrt{p} I_3$ and thus any 3×3 unitary matrix W would satisfy $f = W\Sigma W^T$.

(b) Suppose $p=0$. Then $a=b=c=0$ and $\text{rank}(f) = \text{rank}(\mathbf{0}) = 0$, and thus f has no rank-2 approximation.

Suppose $p \neq 0$ and $q=0$. Then for any orthonormal $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, $\sqrt{p}(\vec{v}_1\vec{v}_1^T + \vec{v}_2\vec{v}_2^T)$ is a rank-2 approximation to f. In particular,

$$\sqrt{p}[(1,0,0)(1,0,0)^T + (0,1,0)(0,1,0)^T] = \sqrt{p}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\sqrt{p}[(1,0,0)(1,0,0)^T + (0,0,1)(0,0,1)^T] = \sqrt{p}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are distinct rank-2 approximations of f.

Suppose $p \neq 0$, $q \neq 0$ and $p-q=0$. Then $f = V\Sigma V^T$ is rank 1 and f has no rank-2 approximation.

Suppose $p \neq 0$, $q > 0$ and $p-q \neq 0$. Then $\sqrt{p+2q} > \sqrt{p-q}$, and thus $\sqrt{p+2q}\vec{u}_3\vec{v}_3^T + \sqrt{p-q}\vec{u}_1\vec{v}_1^T$ and $\sqrt{p+2q}\vec{u}_3\vec{v}_3^T + \sqrt{p-q}\vec{u}_2\vec{v}_2^T$ are distinct rank-2 approximations of f.

Hence for f to have a unique rank-2 approximation, $p \neq 0$, $q < 0$ and $p-q \neq 0$, i.e. a,b,c are not all equal and $ab+bc+ca < 0$.

5. (a) $\tilde{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

$$\begin{aligned} f_{\text{Haar}} &= \tilde{H} f \tilde{H}^T \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 18 & 11 & 19 & 15 \\ 4 & -1 & 5 & 3 \\ -\sqrt{2} & 3\sqrt{2} & 0 & 3\sqrt{2} \\ -5\sqrt{2} & -2\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} = \begin{pmatrix} \frac{63}{4} & -\frac{5}{4} & \frac{7\sqrt{2}}{4} & \sqrt{2} \\ \frac{11}{4} & -\frac{5}{4} & \frac{5\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}. \end{aligned}$$

(b) Since $\tilde{H}$ is unitary, for any $g \in M_{4\times 4}(\mathbb{R})$,

$$\|\tilde{H}^T g \tilde{H} - f\|_F = \|\tilde{H}^T (g - f_{\text{Haar}})\tilde{H}\|_F = \|g - f_{\text{Haar}}\|_F.$$

Hence one should choose to discard the entries with smaller absolute values so as to minimize the Frobenius norm of the difference.

Hence the matrix that should be kept is either $f'_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & 0 \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f'_{\text{Haar}} \tilde{H}$

$$= \frac{1}{16}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & 0 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 84 & 0 & 4\sqrt{2} & 0 \\ 64 & 0 & 20\sqrt{2} & 0 \\ 36 & -12 & -4\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 8\sqrt{2} & 18\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 92 & 76 & 84 & 84 \\ 104 & 24 & 64 & 64 \\ 16 & 32 & 12 & 84 \\ 96 & 64 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{21}{4} & \frac{21}{4} \\ \frac{13}{2} & \frac{3}{2} & 4 & 4 \\ 1 & 2 & \frac{3}{4} & \frac{21}{4} \\ 6 & 4 & \frac{23}{4} & \frac{5}{4} \end{pmatrix};$$

or keep $f''_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & 0 & -\frac{9}{2} \end{pmatrix}$,

whose reconstructed image is given by $\tilde{H}^T f''_{\text{Haar}} \tilde{H}$

$$= \frac{1}{16}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & 0 & -18 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 84 & 0 & 4\sqrt{2} & -6\sqrt{2} \\ 64 & 0 & 20\sqrt{2} & 6\sqrt{2} \\ 36 & -12 & 2\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 2\sqrt{2} & 18\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 92 & 76 & 72 & 96 \\ 104 & 24 & 76 & 52 \\ 28 & 20 & 12 & 84 \\ 84 & 76 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{9}{2} & 6 \\ \frac{13}{2} & \frac{3}{2} & \frac{19}{4} & \frac{13}{4} \\ \frac{7}{4} & \frac{5}{4} & \frac{3}{4} & \frac{21}{4} \\ \frac{21}{4} & \frac{19}{4} & \frac{23}{4} & \frac{5}{4} \end{pmatrix}.$$

6. (a) $\displaystyle\int_{\mathbb{R}} [H_0(t)]^2\,dt = \int_0^1 dt = 1.$

For any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$\int_{\mathbb{R}} [H_{2^p+n}(t)]^2\,dt = \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} (2^{\frac{p}{2}})^2\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})^2\,dt$$

$$= 2 \cdot \frac{1}{2^{p+1}} \cdot 2^p = 1.$$

(b) i. Let $m \in \mathbb{N} \setminus \{0\}$. There exists $p \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$ such that $m = 2^p + n$. Then

$$\begin{aligned}
\langle H_0, H_m \rangle &= \int_{\mathbb{R}} H_0(t) H_{2^p+n}(t)\, dt \\
&= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\, dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})\, dt \\
&= \frac{1}{2^{p+1}} \cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}} \cdot (-2^{\frac{p}{2}}) = 0.
\end{aligned}$$

ii. A. Suppose $p_1 = p_2$. Then

$$\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_1}+n_2}(t)\, dt \\
&= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}} \cdot 0\, dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n+1}{2^{p_1}}} (-2^{\frac{p_1}{2}}) \cdot 0\, dt \\
&+ \int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0 \cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0 \cdot (-2^{\frac{p_1}{2}})\, dt = 0.
\end{aligned}$$

B. Suppose $p_1 < p_2$. Then either

- $2^{p_2-p_1} n_1 \le n_2 < 2^{p_2-p_1}(n_1+0.5)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$; or
- $2^{p_2-p_1}(n_1+0.5) \le n_2 < 2^{p_2-p_1}(n_1+1)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$; or
- $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right) = \varnothing$.

In any case, H_{m_1} is constant on $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$, and thus denoting the constant by c,

$$\begin{aligned}
\langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_2}+n_2}(t)\, dt \\
&= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\, dt + c \int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} (-2^{\frac{p_2}{2}})\, dt \\
&= c[\frac{1}{2^{p_2+1}} \cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}} \cdot (-2^{\frac{p_2}{2}})] = 0.
\end{aligned}$$

7. (a) $\tilde{W} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

$$\begin{aligned}
f_{\text{Walsh}} &= \tilde{W} f \tilde{W}^T \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -2 & 4 & 1 \\ -2 & -1 & -1 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 9 & 0 & 11 & 6 \\ -5 & 0 & -5 & -4 \\ -5 & -6 & -5 & 2 \\ -7 & 2 & -5 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 26 & 8 & -4 & 14 \\ -14 & -4 & 4 & -6 \\ -14 & 8 & -8 & -6 \\ -2 & 8 & -4 & -22 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} & 2 & -1 & \frac{7}{2} \\ -\frac{7}{2} & -1 & 1 & -\frac{3}{2} \\ -\frac{7}{2} & 2 & -2 & -\frac{3}{2} \\ -\frac{1}{2} & 2 & -1 & -\frac{11}{2} \end{pmatrix}.
\end{aligned}$$

(b) The modified Walsh transform f'_{Walsh} is $\begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix}$, whose reconstructed image is given by

$$\begin{aligned}
\tilde{W}^T f'_{\text{Walsh}} \tilde{W} &= \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 14 & 3 & -1 & 2 \\ 8 & 3 & -3 & 10 \\ -2 & 5 & -3 & -6 \\ 8 & -3 & 3 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 14 & 8 & 18 & 16 \\ 18 & -8 & 18 & 4 \\ -10 & -4 & -6 & 12 \\ 18 & 4 & 18 & -8 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{2} & 5 & \frac{9}{2} \\ \frac{9}{2} & -2 & \frac{9}{2} & 1 \\ -\frac{5}{2} & -1 & -\frac{3}{2} & 3 \\ \frac{9}{2} & 1 & \frac{9}{2} & -2 \end{pmatrix}
\end{aligned}$$

8. (a) Note that $W_0 = \mathbf{1}_{[0,1)}$ and thus $(W_0)^2 = \mathbf{1}_{[0,1)}$. Recall that for any $n \in \mathbb{N} \cup \{0\}$, W_n is defined by the recursive relation:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(2t) + (-1)^{j + \lfloor \frac{j}{2} \rfloor} W_j(2t-1)$$

for $j \in \mathbb{N} \cup \{0\}$ and $q \in \{0, 1\}$.
Hence for any $n \in \mathbb{N}$, $(W_n)^2 \equiv \mathbf{1}_{[0,1)}$ and thus

$$\int_{\mathbb{R}} [W_n(t)]^2 \, dt = \int_0^1 \, dt = 1.$$

(b) i. Suppose $j_1 = j_2$. Then $m_1 = 2j_1$ and $m_2 = 2j_1 + 1$, and

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1}(t) W_{2j_1+1}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j_1}{2} \rfloor + 1} W_{j_1}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \cdot (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t-1) \, dt \\
&= -\int_0^1 [W_{j_1}(u)]^2 \, d(\frac{u}{2}) + \int_0^1 [W_{j_1}(v)]^2 \, d(\frac{v-1}{2}) \\
&= -\frac{1}{2} \|W_{j_1}\|^2 + \frac{1}{2} \|W_{j_1}\|^2 = 0.
\end{aligned}$$

ii. Suppose $j_1 < j_2$. Then

$$\begin{aligned}
\langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1+q_1}(t) W_{2j_2+q_2}(t) \, dt \\
&= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor + q_1} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j}{2} \rfloor + q_2} W_{j_2}(2t) \, dt \\
&+ \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2}} W_{j_1}(2t-1) \cdot (-1)^{j_2 + \lfloor \frac{j_2}{2} \rfloor} W_{j_2}(2t-1) \, dt \\
&= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} \cdot \frac{1}{2} \int_0^1 W_{j_1}(u) W_{j_2}(u) \, du \\
&+ (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \cdot \frac{1}{2} \int_0^1 W_{j_1}(v) W_{j_2}(v) \, dv \\
&= \left[(-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} + (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \right] \langle W_{j_1}, W_{j_2} \rangle = 0
\end{aligned}$$

by the induction hypothesis.

Remark. *Recall that $P(m)$ states that*

$$\{W_0, \dots, W_m\} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle).$$

Hence even if we have proven $P(m)$ to be true for any $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{W} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$$

has not been directly proven. The subtle difference is easier to observe if we consider the statements

$$\tilde{P}(m) : \{0, \dots, m\} \text{ is finite}$$

and

$$\mathbb{N} \cup \{0\} \text{ is finite,}$$

for which $\tilde{P}(m)$ being true for any $m \in \mathbb{N} \cup \{0\}$ does not imply the truthfulness of the second statement. However, since the orthogonality of $\mathcal{W}$ depends on the orthogonality of pairs of its elements, and each pair of its elements is contained in some $\{W_0 \dots, W_m\}$, the induction result suffices.

9. (a) $W_0 = \mathbf{1}_{[0,1)}$, so $W_0(0) = \lim_{t \to 1^-} W_0(t) = 1$.
   Let $P(k)$ be the proposition that

$$\lim_{t \to 1^-} W_k(t) = \begin{cases} W_k(0) & \text{if } k \text{ is even,} \\ -W_k(0) & \text{if } k \text{ is odd.} \end{cases}$$

   Suppose $P(j)$ is true for some $j \in \mathbb{N} \cup \{0\}$.
   Then for $q \in \{0, 1\}$,

$$\begin{aligned} \lim_{t \to 1^-} W_{2j+q}(t) &= \lim_{t \to 1^-} (-1)^{j+q} W_j(2t-1) \\ &= (-1)^{j+q} \lim_{t \to 1^-} W_j(t) \\ &= \begin{cases} (-1)^q \lim_{t \to 1^-} W_j(t) & \text{if } j \text{ is even,} \\ (-1)^{q+1} \lim_{t \to 1^-} W_j(t) & \text{if } j \text{ is odd} \end{cases} \\ &= (-1)^q W_j(0) = \begin{cases} W_j(0) & \text{if } q = 0, \\ -W_j(0) & \text{if } q = 1. \end{cases} \end{aligned}$$

   Hence $P(2j)$ and $P(2j+1)$ are also true. By induction, $P(k)$ is true for any $k \in \mathbb{N} \cup \{0\}$.

   (b) $W_0 \equiv \mathbf{1}_{[0,1)}$ and thus has 0 zero-crossings on (0,1).
   Let $P(k)$ be the proposition that W_k has k zero-crossings on $(0, 1)$.
   Suppose $P(j)$ is true for some $j \in \mathbb{N} \cup \{0\}$. Then a direct observation is that both W_{2j} and W_{2j+1} have j zero-crossings on each of $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$; it remains to verify whether $\frac{1}{2}$ is a zero-crossing of theirs.
   For $q \in \{0, 1\}$,

$$\begin{aligned} \lim_{t \to \frac{1}{2}^-} W_{2j+q}(t) &= (-1)^{\lfloor \frac{j}{2} \rfloor + q} \lim_{t \to 1^-} W_j(t) \\ &= \begin{cases} (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(0) & \text{if } j \text{ is even,} \\ (-1)^{\lfloor \frac{j}{2} \rfloor + q + 1} W_j(0) & \text{if } j \text{ is odd} \end{cases} \\ &= \begin{cases} (-1)^{j+q} W_{2j+q}(\frac{1}{2}) & \text{if } j \text{ is even,} \\ (-1)^{j+q+1} W_{2j+q}(\frac{1}{2}) & \text{if } j \text{ is odd} \end{cases} \\ &= (-1)^q W_{2j+q}\left(\frac{1}{2}\right) = \begin{cases} \lim_{t \to \frac{1}{2}^+} W_{2j+q}(t) & \text{if } q = 0, \\ -\lim_{t \to \frac{1}{2}^+} W_{2j+q}(t) & \text{if } q = 1. \end{cases} \end{aligned}$$

   Hence $\frac{1}{2}$ is a zero-crossing of W_{2j+q} if $q = 1$ and is not if $q = 0$, and thus W_{2j} has $2j$ zero-crossings on (0,1) while W_{2j+1} has $2j + 1$.
   By induction, $P(k)$ is true for any $k \in \mathbb{N} \cup \{0\}$.

10. (a) The idea is basically the same with Q6(b)(ii)(B):

Let $\varnothing \neq M \subseteq \mathbb{N} \setminus \{0\}$ and let $m^* = \max M$.

Then $\prod_{m \in M \setminus \{m^*\}} R_m$ is constant on each period of R_{m^*}, i.e.

$$\prod_{m \in M \setminus \{m^*\}} R_m \equiv c_k \in \{\pm 1\} \text{ on each } \left(\frac{k}{2^{m^*-1}}, \frac{k+1}{2^{m^*-1}}\right), k = 0, 1, \cdots, 2^{m^*-1} - 1.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}} \prod_{m \in M} R_m &= \sum_{k=0}^{2^{m^*}-1} \int_{\frac{k}{2^{m^*-1}}}^{\frac{k+1}{2^{m^*-1}}} c_k R_{m^*}(t)\, dt \\ &= \sum_{k=0}^{2^{m^*}-1} c_k \left(\int_{\frac{2k}{2^{m^*}}}^{\frac{2k+1}{2^{m^*}}} dt - \int_{\frac{2k+1}{2^{m^*}}}^{\frac{2k+2}{2^{m^*}}} dt\right) = 0.\end{aligned}$$

(b) Let M be a finite subset of $\mathbb{N} \setminus \{0\}$.

For any $m \in \mathbb{N}\setminus\{0\}$, ${R_m}^2 \equiv \mathbf{1}_{[0,1)}$ except at finitely many points. Hence $\left(\prod_{m \in M} R_m\right)^2 \equiv \mathbf{1}_{[0,1)}$ except at finitely many points, and

$$\int_{\mathbb{R}} \left(\prod_{m \in M} R_m\right)^2 = \int_0^1 dt = 1.$$

Let M_1 and M_2 be distinct finite subsets of $\mathbb{N} \setminus \{0\}$. Then $(M_1 \setminus M_2) \sqcup (M_2 \setminus M_1) \neq \varnothing$. Except at finitely many points,

$$\begin{aligned}\prod_{m \in M_1} R_m \cdot \prod_{n \in M_2} R_n &= \prod_{m \in M_1 \cap M_2} R_m^2 \cdot \prod_{n \in M_1 \setminus M_2} R_n \cdot \prod_{p \in M_2 \setminus M_1} R_p \\ &= \prod_{m \in (M_1 \setminus M_2) \sqcup (M_2 \setminus M_1)} R_m,\end{aligned}$$

and thus by the result of (a),

$$\int_{\mathbb{R}} (\prod_{m \in M_1} R_m \cdot \prod_{n \in M_2} R_n) = \int_{\mathbb{R}} \prod_{m \in (M_1 \setminus M_2) \sqcup (M_2 \setminus M_1)} R_m = 0.$$

11. (a) $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

$$\begin{aligned}\hat{f} &= UfU \\ &= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 9 & -1 & 10 & 5 \\ 5 & 3+4j & 6 & 1-2j \\ -7 & 3 & -6 & 9 \\ 5 & 3-4j & 6 & 1+2j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 23 & -1+6j & 15 & -1-6j \\ 15+2j & 5-2j & 7-2j & -7+2j \\ -1 & -1+6j & -25 & -1-6j \\ 15-2j & -7-2j & 7+2j & 5+2j \end{pmatrix}.\end{aligned}$$

(b) The submatrix of $\hat{f}$ formed by the three frequencies closest to 0 is

$$\hat{f}' = \frac{1}{16}\begin{pmatrix} 23 & 1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix},$$

whose reconstructed image is

$$\begin{aligned}
(4\bar{U})\hat{f}'(4\bar{U}) &= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 23 & 1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 53 & -3+2j & 0 & -3-2j \\ 19 & -1+18j & 0 & -1-18j \\ -7 & 1+10j & 0 & 1-10j \\ 27 & -1-6j & 0 & -1+6j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 47 & 49 & 59 & 57 \\ 17 & -17 & 21 & 55 \\ -5 & -27 & -9 & 13 \\ 25 & 39 & 29 & 15 \end{pmatrix}.
\end{aligned}$$

12. (a) i. Refer to DFT of convolution of **Further properties of DFT** in Section 4.

ii.

$$\begin{aligned}
iDFT(MN\hat{f}\odot\hat{g})(k,l) &= MN\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m,n)\hat{g}(m,n)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \frac{1}{MN}\sum_{m,k',k''=0}^{M-1}\sum_{n,l',l''=0}^{N-1} f(k',l')g(k'',l'')e^{2\pi j(\frac{m(k-k'-k'')}{M}+\frac{n(l-l'-l'')}{N})} \\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')\mathbf{1}_{M\mathbb{Z}}(k-k'-k'')\mathbf{1}_{N\mathbb{Z}}(l-l'-l'') \\
&= \sum_{k',k''=0}^{M-1}\sum_{l',l''=0}^{N-1} f(k',l')g(k'',l'')[\delta(k-k'-k'')+\delta(k-k'-k''+M)] \\
&\quad [\delta(l-l'-l'')+\delta(l-l'-l''+N)] \\
&= \sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1} f(k',l')g(k-k',l-l') = f*g(k,l).
\end{aligned}$$

(b) Withheld until the due date of Assignment 3.

(c) i.

$$\begin{aligned}
\hat{\tilde{f}}(m,n) &= \frac{1}{N^2}\sum_{k,l=0}^{N-1}\tilde{f}(k,l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(l,-k)e^{-2\pi j\frac{mk+nl}{N}},
\end{aligned}$$

whereas

$$
\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(n,-m) \\
&= \frac{1}{N^2}\sum_{k,l=0}^{N-1} f(k,l)e^{-2\pi j\frac{nk-ml}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\sum_{k'=1-N}^{0} f(l',-k')e^{-2\pi j\frac{mk'+nl'}{N}} \\
&= \frac{1}{N^2}\sum_{l'=0}^{N-1}\left(f(l',0)e^{-2\pi j\frac{nl'}{N}} + \sum_{k'=1-N}^{0} f(l',-k')e^{-2\pi j\frac{mk'+nl'}{N}}\right) \\
&= \frac{1}{N^2}\sum_{k',l'=0}^{N-1} f(l',-k')e^{-2\pi j\frac{mk'+nl'}{N}} = \hat{\tilde{f}}(m,n).
\end{aligned}
$$

ii.

$$
\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m,n=0}^{N-1} \tilde{\hat{f}}(m,n)e^{2\pi j\frac{mk+nl}{N}} \\
&= \sum_{m,n=0}^{N-1} \hat{f}(n,-m)e^{2\pi j\frac{mk+nl}{N}} \\
&= \sum_{m'=0}^{N-1}\sum_{n'=1-N}^{0} \hat{f}(m',n')e^{2\pi j\frac{-n'k+m'l}{N}} \\
&= \sum_{m'=0}^{N-1}\left(\hat{f}(m',0)e^{2\pi j\frac{m'l}{N}} + \sum_{n'=1-N}^{-1} \hat{f}(m',n'+N)e^{2\pi j\frac{-n'k+m'l}{N}}\right) \\
&= \sum_{m',n'=0}^{N-1} \hat{f}(m',n')e^{2\pi j\frac{m'l-n'k}{N}} = f(l,-k).
\end{aligned}
$$

(d) WLOG assume $k_0 \in \mathbb{Z} \cap [0, M-1]$ and $l_0 \in \mathbb{Z} \cap [0, N-1]$.

i. Refer to DFT of a shifted image of **Further properties of DFT** in Section 4.

ii.

$$
\begin{aligned}
iDFT(e^{-2\pi j(\frac{k_0 m}{M}+\frac{l_0 n}{N})}\hat{f})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}(m,n)e^{2\pi j(\frac{m(k-k_0)}{M}+\frac{n(l-l_0)}{N})} \\
&= f(k-k_0, l-l_0).
\end{aligned}
$$

(e) i.

$$
\begin{aligned}
\tilde{\hat{f}}(m,n) &= \hat{f}(m-m_0, n-n_0) \\
&= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)e^{-2\pi j(\frac{k(m-m_0)}{M}+\frac{l(n-n_0)}{N})} \\
&= DFT(e^{2\pi j(\frac{m_0 k}{M}+\frac{n_0 l}{N})}f)(m,n).
\end{aligned}
$$

ii.

$$
\begin{aligned}
iDFT(\tilde{\hat{f}})(k,l) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \tilde{\hat{f}}(m,n)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}(m-m_0, n-n_0)e^{2\pi j(\frac{mk}{M}+\frac{nl}{N})} \\
&= \sum_{m'=-m_0}^{M-1-m_0}\sum_{n'=-n_0}^{N-1-n_0} \hat{f}(m',n')e^{2\pi j(\frac{(m'+m_0)k}{M}+\frac{(n'+n_0)l}{N})} \\
&= e^{2\pi j(\frac{m_0 k}{M}+\frac{n_0 l}{N})}f(k,l).
\end{aligned}
$$

13. (a) Note that for any $c \in \mathbb{Z}$, $\cos \pi k = (-1)^k$ and thus

$$\cos \pi k + \cos(-\pi k) = (-1)^k + (-1)^{-k} = 0.$$

$$\begin{aligned}
&iEDCT(EDCT(f))(k,l) \\
&= \frac{1}{MN} \sum_{m,k'=0}^{M-1} \sum_{n,l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] f(k',l') \\
&\quad \cos \frac{\pi m(2k+1)}{2M} \cos \frac{\pi m(2k'+1)}{2M} \cos \frac{\pi n(2l+1)}{2N} \cos \frac{\pi n(2l'+1)}{2N} \\
&= \frac{1}{4MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m'=1-M}^{M-1} \sum_{n'=1-N}^{N-1} f(k',l') \\
&\quad \left(\cos \frac{\pi m(k+k'+1)}{M} + \cos \frac{\pi m(k-k')}{M} \right) \left(\cos \frac{\pi n(l+l'+1)}{N} + \cos \frac{\pi n(l-l')}{N} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{16MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m'=-M}^{M-1} \sum_{n'=-N}^{N-1} f(k',l') \\
&\quad \left(e^{2\pi j \frac{m(k+k'+1)}{2M}} + e^{-2\pi j \frac{m(k+k'+1)}{2M}} + e^{2\pi j \frac{m(k-k')}{2M}} + e^{-2\pi j \frac{m(k-k')}{2M}} \right) \\
&\quad \left(e^{2\pi j \frac{n(l+l'+1)}{2N}} + e^{-2\pi j \frac{n(l+l'+1)}{2N}} + e^{2\pi j \frac{n(l-l')}{2N}} + e^{-2\pi j \frac{n(l-l')}{2N}} \right) \\
&= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k',l') [\mathbf{1}_{2M\mathbb{Z}}(k+k'+1) + \mathbf{1}_{2M\mathbb{Z}}(k-k')][\mathbf{1}_{2N\mathbb{Z}}(l+l'+1) + \mathbf{1}_{2N\mathbb{Z}}(l-l')] \\
&= f(k,l).
\end{aligned}$$

(b) Assuming f is symmetrically extended about $x = 0$ and $y = 0$,

$$\begin{aligned}
&iODCT(ODCT(f))(k,l) \\
&= \frac{1}{(2M-1)(2N-1)} \sum_{m,k'=0}^{M-1} \sum_{n,l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(k)][2 - \mathbf{1}_{N\mathbb{Z}}(l)][2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] \\
&\quad f(k',l') \cos \frac{2\pi mk}{2M-1} \cos \frac{2\pi mk'}{2M-1} \cos \frac{2\pi nl}{2N-1} \cos \frac{2\pi nl'}{2N-1} \\
&= \frac{1}{4(2M-1)(2N-1)} \sum_{m,k'=1-M}^{M-1} \sum_{n,l'=1-N}^{N-1} f(k',l') \\
&\quad \left(\cos \frac{2\pi m(k+k')}{2M-1} + \cos \frac{2\pi m(k-k')}{2M-1} \right) \left(\cos \frac{2\pi n(l+l')}{2N-1} + \cos \frac{2\pi n(l-l')}{2N-1} \right) \\
&= \frac{1}{16(2M-1)(2N-1)} \sum_{m,k'=1-M}^{M-1} \sum_{n,l'=1-N}^{N-1} f(k',l') \\
&\quad \left(e^{2\pi j \frac{m(k+k')}{2M-1}} + e^{-2\pi j \frac{m(k+k')}{2M-1}} + e^{2\pi j \frac{m(k-k')}{2M-1}} + e^{-2\pi j \frac{m(k-k')}{2M-1}} \right) \\
&\quad \left(e^{2\pi j \frac{n(l+l')}{2N-1}} + e^{-2\pi j \frac{n(l+l')}{2N-1}} + e^{2\pi j \frac{n(l-l')}{2N-1}} + e^{-2\pi j \frac{n(l-l')}{2N-1}} \right) \\
&= \frac{1}{4} \sum_{k'=1-M}^{M-1} \sum_{l'=1-N}^{N-1} f(k',l') \\
&\quad [\mathbf{1}_{(2M-1)\mathbb{Z}}(k+k') + \mathbf{1}_{(2M-1)\mathbb{Z}}(k-k')][\mathbf{1}_{(2N-1)\mathbb{Z}}(l+l') + \mathbf{1}_{(2N-1)\mathbb{Z}}(l-l')] \\
&= \frac{1}{4}[f(-k,-l) + f(k,-l) + f(-k,l) + f(k,l)] = f(k,l).
\end{aligned}$$

(c) Note that for any $c \in \mathbb{Z}$, $\sin c\pi = 0$.

$$iEDST(EDST(f))(k,l)$$
$$= \frac{1}{MN} \sum_{m,k'=0}^{M-1} \sum_{n,l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] f(k',l')$$
$$\sin \frac{\pi m(2k+1)}{2M} \sin \frac{\pi m(2k'+1)}{2M} \sin \frac{\pi n(2l+1)}{2N} \sin \frac{\pi n(2l'+1)}{2N}$$
$$= \frac{1}{4MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} f(k',l')$$
$$\left(\cos \frac{\pi m(k+k'+1)}{M} - \cos \frac{\pi m(k-k')}{M}\right)\left(\cos \frac{\pi n(l+l'+1)}{N} - \cos \frac{\pi n(l-l')}{N}\right)$$
$$= \frac{1}{16MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} f(k',l')$$
$$\left(e^{2\pi j \frac{m(k+k'+1)}{2M}} + e^{-2\pi j \frac{m(k+k'+1)}{2M}} - e^{2\pi j \frac{m(k-k')}{2M}} - e^{-2\pi j \frac{m(k-k')}{2M}}\right)$$
$$\left(e^{2\pi j \frac{n(l+l'+1)}{2N}} + e^{-2\pi j \frac{n(l+l'+1)}{2N}} - e^{2\pi j \frac{n(l-l')}{2N}} - e^{-2\pi j \frac{n(l-l')}{2N}}\right)$$
$$= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k',l')[\mathbf{1}_{2M\mathbb{Z}}(k+k'+1) - \mathbf{1}_{2M\mathbb{Z}}(k-k')][\mathbf{1}_{2N\mathbb{Z}}(l+l'+1) - \mathbf{1}_{2N\mathbb{Z}}(l-l')]$$
$$= f(k,l).$$

(d)

$$iODST(ODST(f))(k,l)$$
$$= \frac{16}{(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=1}^{M} \sum_{n=1}^{N} f(k',l')$$
$$\sin \frac{2\pi m(k+1)}{2M+1} \sin \frac{2\pi m(k'+1)}{2M+1} \sin \frac{2\pi n(l+1)}{2N+1} \sin \frac{2\pi n(l'+1)}{2N+1}$$
$$= \frac{1}{(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^{M} \sum_{n=-N}^{N} f(k',l')$$
$$\left(\cos \frac{2\pi m(k+k'+2)}{2M+1} - \cos \frac{2\pi m(k-k')}{2M+1}\right)\left(\cos \frac{2\pi n(l+l'+2)}{2N+1} - \cos \frac{2\pi n(l-l')}{2N+1}\right) = \frac{1}{4(2M+1)(2N+}$$
$$\left(e^{2\pi j \frac{m(k+k'+2)}{2M+1}} + e^{-2\pi j \frac{m(k+k'+2)}{2M+1}} - e^{2\pi j \frac{m(k-k')}{2M+1}} - e^{-2\pi j \frac{m(k-k')}{2M+1}}\right)$$
$$\left(e^{2\pi j \frac{n(l+l'+2)}{2N+1}} + e^{-2\pi j \frac{n(l+l'+2)}{2N+1}} - e^{2\pi j \frac{n(l-l')}{2N+1}} - e^{-2\pi j \frac{n(l-l')}{2N+1}}\right)$$
$$= \frac{1}{4(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k',l')$$
$$[2(2M+1)\mathbf{1}_{(2M+1)\mathbb{Z}}(k+k'+2) - 2(2M+1)\mathbf{1}_{(2M+1)\mathbb{Z}}(k-k')]$$
$$[2(2N+1)\mathbf{1}_{(2N+1)\mathbb{Z}}(l+l'+2) - 2(2N+1)\mathbf{1}_{(2N+1)\mathbb{Z}}(l-l')]$$
$$= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k',l')\delta(k-k')\delta(l-l') = f(k,l).$$