MATH3310 Midterm practice

1 Integrating factors

Let $y = y(x)$. Use the integrating factor method, solve

1. $y' + y = x$, $x > 0$ with $y(0) = 2$;
2. $y' + y = e^{-x}$, $x > 0$ with $y(0) = 1$;
3. $xy' + 2y = 10x^2$, $x > 0$ with $y(0) = 3$;
4. $y'' + y' - 6y = 0$ with $y(0) = 1, y(1) = 2$;
5. $-2y'' + 3y = x^2 + 1$ with $y'(0) = 0, y(1) = 1$.

2 Spectral method for PDE

Solve the Following PDEs

1. $$\begin{cases} \frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty) \times (0, 2\pi) \\ u(0,x) = 20 & x \in (0, 2\pi) \\ u(t,0) = u(t,2\pi) = 0 & t \in [0, \infty) \end{cases}$$

2. $$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty) \times (0, \pi) \\ u(0,x) = \sin x & x \in (0, 2\pi) \\ u(t,0) = u(t,\pi) = 0 & t \in [0, \infty) \end{cases}$$

3. $\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty) \times (0,L) \\ u(0,x) = \sin\frac{2\pi}{L}x + \cos\frac{2\pi}{L}x & x \in (0,L) \\ u(t,0) = u(t,L), \quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) & t \in [0,\infty) \end{cases}$

4. $\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty) \times (0,2\pi) \\ u(0,x) = u_0(x) & x \in (0,2\pi) \\ \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,2\pi) = 0 & t \in [0,\infty) \end{cases}$.

3 Computing Fourier series

Compute the complex Fourier series of the following 2π-periodic functions

1. $f(x) = -|x| + \pi$, $-\pi \le x \le \pi$

2. $f(x) = \begin{cases} \pi & -\pi \le x \le 0 \\ x & 0 < x \le \pi \end{cases}$

3. $f(x) = \begin{cases} -x - \pi & -\pi \le x \le 0 \\ -x + \pi & 0 < x \le \pi \end{cases}$

4 Computing Fourier transform

Compute the Fourier transforms of the following functions

1. $f(x) = \frac{1}{a^2+x^2}$;

2. $f(x) = e^{-a|x|}$;

3. $f(x) = \begin{cases} 0 & |x| > a \\ |x| & \text{otherwise} \end{cases}$.

5 Discrete Fourier transform and numerical PDE

1. Consider the PDE: $\frac{d^2u}{dx^2} = xe^x$ for $x \in [0, 2\pi]$ with periodic boundary condition. Divide the interval $[0, 2\pi]$ using 9 points: $x_0, \ldots, x_8$. Approximate $\frac{d^2}{dx^2}$ by central difference approximation. Use the spectral method to approximate: $u_0 = u(x_0), ..., u_8 = u(x_8)$.

2. Let $p_1^{(0)}, \ldots, p_n^{(0)}$ be n points in $\mathbb{R}^N$, given in such an order. Assume further that their mass is centered at the origin

$$\frac{1}{n}\sum_{i=1}^{n} p_i^{(0)} = 0.$$

Each time we compute the midpoint of the two neighboring points,

$$p_i^{(k)} = \frac{1}{2}\left(p_i^{(k-1)} + p_{i+1}^{(k-1)}\right)$$

and we consider $p_1^{(k)}$ and $p_n^{(k)}$ to be also neighboring points. Show that these n points converge to the origin.

3. The **auto-correlation** of a vector $\mathbf{v} \in \mathbb{C}^N$ is defined to be

$$\mathbf{r} = T_{\bar{\mathbf{v}}}\mathbf{v}$$

where $T_{\bar{\mathbf{v}}}$ is defined as before, $\bar{\mathbf{v}}$ is the entry-wise complex conjugation of $\mathbf{v}$. What are the discrete Fourier coefficients of $\mathbf{r}$?

6 Iterative methods

1. The Jacobi iteration for a general 2×2 matrix has

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, M = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Find the eigenvalues of $B = M^{-1}(M - A)$. If A is symmetric and positive definite, show that the iteration converges.

2. For the Gauss-Seidel the matrices are

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Find the eigenvalues of $B = M^{-1}(M - A)$. Give an example of a matrix A for which the Gauss-Seidel iteration will NOT converge.

3. Decide the convergence or divergence of Jacobi and Gauss-Seidel method iterations for
$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$
Construct M for both methods and find the eigenvalues of $B = I - M^{-1}A$.

7 Questions that help you understand the course materials

1. In this exercise we show how the symmetries of a function imply certain properties of its Fourier series. Let $f \in C([-\pi,\pi],\mathbb{C})$, and
$$\hat{f}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x)e^{-inx}dx$$
   (a) Show that the Fourier series of the function f can be written as
$$\hat{f}(0) + \sum_{n=1}^{\infty}[\hat{f}(n)+\hat{f}(-n)]\cos\theta + i[\hat{f}(n)-\hat{f}(-n)]\sin\theta.$$
   (b) Show that if f is even, then $\hat{f}(n) - \hat{f}(-n) = 0$, so we get a cosine series (with possibly complex coefficients).
   (c) Show that if f is odd, then $\hat{f}(n) + \hat{f}(-n) = 0$, so we get a sine series (with possibly complex coefficients).
   (d) Show that $f : [-\pi,\pi] \to \mathbb{R}$, i.e. real valued, if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$. So the coefficients of cosines and sines are real. Because of this property, if f is real-valued, sometimes we call
$$\hat{f}(0) + \sum_{n=1}^{\infty}[\hat{f}(n)+\hat{f}(-n)]\cos\theta + i[\hat{f}(n)-\hat{f}(-n)]\sin\theta$$
   the real Fourier series, and
$$\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}$$

the complex Fourier series. They are seen to be equivalent expressions for $f \in C([-\pi, \pi], \mathbb{R})$.

2. Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are continuous and 2π-periodic. Then

$$\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n), \quad n \in \mathbb{Z}$$

3. Show that

   (a) for a function $f : [0, 2\pi] \to \mathbb{C}$, its zero-th Fourier coefficient $\hat{f}(0)$ is the average of the function f up to dividing by 2π

   $$\hat{f}(0) = \frac{1}{2\pi} \int_{[0,2\pi]} f(x)dx.$$

   (b) For a function $f : \mathbb{R} \to \mathbb{C}$, its Fourier transform evaluated at 0 is the average of the function

   $$\hat{f}(0) = \int_{\mathbb{R}} f(x)dx.$$

4. What is the matrix for the central difference scheme with homogeneous Dirichlet boundary condition? Can you still diagonalize it with discrete Fourier transform?

5. What is the matrix for the central difference scheme with homogeneous Neumann boundary condition? Can you still diagonalize it with discrete Fourier transform?