Lecture 7:

Recap: Analytic Spectral (Fourier) method

1. For bounded domain $[0, 2\pi]$, we introduced <u>Fourier Series method</u>.

   Write $u(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$

   Eigenvectors of some differential operators (such as $\frac{d^2}{dx^2}$)

   $\downarrow$

   Kill differential operator

   $\downarrow$

   Algebraic equation (or ODE)
   (by comparing coefficients)

Example: Consider $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \sin x + 2\cos x$ where $y(0) = y(2\pi)$ (periodic). $(y : [0, 2\pi] \to \mathbb{R})$

Write: $y(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$

$$\sum_{n=1}^{N} \left[(n b_n - n^2 a_n) \cos nx - (n a_n + n^2 b_n) \sin nx \right] = \sin x + 2\cos x$$

Compare coefficients

$$b_1 - a_1 = 2$$

$$a_1 + b_1 = -1$$

Example:

Consider: $u_t = u_{xx}$, $x \in [0, 2\pi]$ such that

$$u(0,t) = u(2\pi, t) \quad \text{(periodic)}$$

$$u(x,0) = f(x) \quad \text{(initial condition)}$$

$(u : \underset{x}{[0,2\pi]} \times \underset{t}{\mathbb{R}} \to \mathbb{R})$

Write: $u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos nx + b_n(t) \sin nx$

$$\Downarrow$$

$$\sum_{n=0}^{N} a_n'(t) \cos nx + b_n'(t) \sin nx = \sum_{n=0}^{N} (-n^2) a_n(t) \cos nx + (-n^2) b_n(t) \sin nx$$

$\Downarrow$ Comparing coefficients

$$a_n'(t) = -n^2 a_n(t) \quad \text{and} \quad b_n'(t) = -n^2 b_n(t) \quad \text{(ODE)}$$

2. For unbounded domain $(-\infty, \infty)$, we introduced the Fourier transform method.

Example: Consider the ODE: $\frac{dy}{dt} - 4y = H(t)e^{-4t}$ where $t \in \mathbb{R}$

and $H(t)$ is given by: $H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$

Apply Fourier Transform on both sides:

$$\widehat{\frac{dy}{dt}}(k) - 4\hat{y}(k) = \widehat{H(t)e^{-4t}}(k) \left(= \frac{1}{4+ik} \right)$$

$$(ik-4)\hat{y}(k) = \frac{1}{4+ik} \quad \text{(Algebraic eqt)}$$

Example: We consider: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ where $c \neq 0$

($u = u(x, t)$ is a function defined on $-\infty < x < \infty$ $t \geq 0$)

Apply Fourier Transform w.r.t. x on both sides:

$$\frac{d^2}{dt^2} \hat{u}(k, t) = -c^2 k^2 \hat{u}(k, t)$$

(ODE if fixing k)

Question: Extension to discrete case (Computational Math.)

Answer: Discrete Fourier Transform

Goal: ① Define discrete Fourier Transform (DFT)

② Use DFT to solve discretized differential eqt.

Definition: (Discrete Fourier Transform) Given $f_0, f_1, \ldots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

The inverse discrete Fourier Transform recovers the original signal:

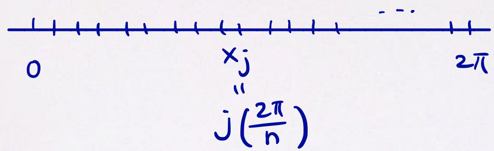
$$f_j = \sum_{k=0}^{n-1} c_k e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \ldots, n-1$$

Motivation of the definition :

Let $f(x)$ be defined on $[0, 2\pi]$. Approximate $f(x)$ by :

$$F_n(x) = \sum_{k=0}^{n-1} c_k e^{ikx}, \quad x \in [0, 2\pi] \quad \text{such that :}$$

$$F_n(x_j) = f(x_j) := f_j, \quad x_j = j\left(\frac{2\pi}{n}\right)$$



Then, we have :

$$(*)\begin{cases} F_n(\overset{0}{\overset{\shortparallel}{x_0}}) = c_0 + c_1 + c_2 + \dots + c_{n-1} = f_0 \\ F_n(x_1) = c_0 + c_1 e^{ix_1} + c_2 e^{i2x_1} + \dots + c_{n-1} e^{(n-1)ix_1} = f_1 \\ \vdots \\ F_n(x_{n-1}) = c_0 + c_1 e^{ix_{n-1}} + \dots + c_{n-1} e^{(n-1)ix_{n-1}} = f_{n-1} \end{cases}$$

Let $\omega = e^{\frac{2\pi i}{n}} = e^{ix_1}$. Then: $\omega^2 = e^{\frac{4\pi i}{n}} = e^{ix_2}$, $\omega^3 = e^{ix_3}$ etc...

$\therefore$ (*) can be written as:

$$A_\omega = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

Note: $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \dfrac{1-\omega^n}{1-\omega} = 0 \quad \left(\because \left(e^{\frac{2\pi i}{n}}\right)^n = \omega^n = 1\right)$

$$\therefore (A_\omega \overline{A_\omega})_{j,k} = 1\cdot 1 + \omega^j \overline{\omega}^k + \omega^{2j}\overline{\omega}^{2k} + \cdots + \omega^{(n-1)j}\overline{\omega}^{(n-1)k}$$

$$= 1 + e^{2\pi i\frac{(j-k)}{n}} + \cdots + e^{2\pi i\frac{(n-1)(j-k)}{n}} = \begin{cases} n & \text{if } j=k \\ \dfrac{1-e^{2\pi i(j-k)}}{1-e^{2\pi i\frac{(j-k)}{n}}} = 0 & \text{if } j\neq k. \end{cases}$$

$$\therefore \quad A_\omega \overline{A_\omega} = \overline{A_\omega} A_\omega = n I .$$

We have:
$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = A_\omega^{-1} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} = \frac{\overline{A_\omega}}{n} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

Hence,
$$c_k = \frac{1}{n}\left(f_0 + e^{-\frac{2\pi i}{n} k} f_1 + e^{-\frac{4\pi i}{n} k} f_2 + \cdots + e^{-\frac{2\pi i}{n} k(n-1)} f_{n-1} \right)$$

$$k = 0, 1, 2, \ldots, n-1$$

"

DFT !!

Remark: Computational cost for DFT is:

$$\left.\begin{array}{c} n^2 \text{ multiplication} \\ + \\ n(n-1) \text{ addition} \end{array}\right\} = O(n^2)$$

Example: Consider $f(t) = 5 + 2\cos(t - \frac{\pi}{2}) + 3\cos(2t)$.

f is 2π-periodic. Divide $[0, 2\pi]$ by 4 partitions. Find the DFT of f (discretized by 4 points).

$$f_0 = f(0) = 8 \; ; \; f_1 = f\left(\frac{2\pi}{4}\right) = 4 \; ; \; f_2 = f\left(\frac{4\pi}{4}\right) = 8 \; ; \; f_3 = f\left(\frac{6\pi}{4}\right) = 0$$

$\therefore$ DFT: $$c_k = \frac{1}{4}\sum_{j=0}^{3} f_j \, e^{-i\left(\frac{2jk\pi}{4}\right)} \quad \text{for } k = 0, 1, 2, 3 \quad \text{or}$$

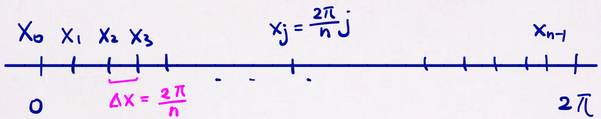
$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{\overline{A_w}}{\underset{\overset{\shortparallel}{4}}{n}} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}}{4} \begin{pmatrix} 8 \\ 4 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -i \\ 3 \\ i \end{pmatrix}$$

Another interpretation of DFT

Recall: Fourier Transform is the extension of Fourier series to $(-\infty, \infty)$.

Fourier coefficient is given by: $c_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \underbrace{f(x)}_{2\pi\text{-periodic}}\, e^{-ikx}\, dx$

Divide:



We can approximate the integration by:

$$c_k \approx \frac{1}{2\pi}\sum_{j=0}^{n-1} f(x_j)\, e^{-i x_j k}\, \Delta x = \frac{1}{\cancel{2\pi}}\sum_{j=0}^{n-1} f(x_j)\, e^{-i\frac{2\pi}{n} jk}\left(\frac{\cancel{2\pi}}{n}\right)$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} f_j\, e^{-i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

$\therefore$ DFT = approximation of (complex) Fourier coefficient.

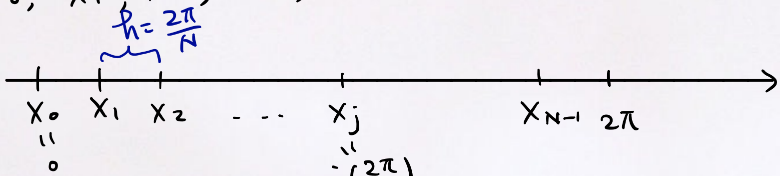
DFT and numerical differential equation

Consider: $\dfrac{d^2 u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition.

$u(0) = u(2\pi)$

Suppose f is measured only at N discrete points :

$x_0, x_1, x_2, \ldots, x_{N-1}$



(unknown)

Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

By Taylor's expansion,

$$u(x_j + h) \approx u(x_j) + h u'(x_j) + \frac{h^2}{2} u''(x_j) \quad \text{—— ①}$$

$$u(x_j - h) \approx u(x_j) - h u'(x_j) + \frac{h^2}{2} u''(x_j) \quad \text{—— ②}$$

① + ② : $$u''(x_j) \approx \frac{u(\underbrace{x_j - h}_{= x_{j-1}}) - 2u(x_j) + u(\underbrace{x_j + h}_{= x_{j+1}})}{h^2}$$

(Central difference approximation of u'')

$$\therefore \quad u''(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \quad \text{where} \quad h = \frac{2\pi}{N}$$

Thus, $$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & 0 & 1 \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \in M_{N\times N}(\mathbb{R})$$

(using the fact $u_0 = u_N$, $u_{-1} = u_{N-1}$)

Thus, $\frac{d^2 u}{dx^2} = f$ can be discretized as $\boxed{\tilde{D}\,\vec{u} = \vec{f}}$ (Linear System)

Remark: Note that $\tilde{D}$ can be a very BIG matrix.

Goal: Design the numerical spectral method. We need to:

① Determine eigenvalues / eigenvectors of $\tilde{D}$

② Rank of $\tilde{D}$ (to understand the sol. of linear system)

In continuous case, e^{ikx} is an eigenfunction of $\frac{d^2}{dx^2}$, that is periodic.

In discrete case, define:

$$\underset{\displaystyle \mathbb{C}^N}{\overrightarrow{e^{ikx}}} \overset{\text{def}}{=\!=} \begin{pmatrix} e^{ikx_0} \\ e^{ikx_1} \\ \vdots \\ e^{ikx_{N-1}} \end{pmatrix}$$

(Capture the values of e^{ikx} at N discrete points)

Claim: $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$

Proof: For each j, $\left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_j = \dfrac{e^{ikx_{j-1}} - 2e^{ikx_j} + e^{ikx_{j+1}}}{h^2}$ $\quad x_{j-1} = x_j - h,\quad x_{j+1} = x_j + h$

$$= e^{ikx_j}\left(\frac{e^{-ikh} - 2 + e^{ikh}}{h^2}\right)$$

$$= e^{ikx_j}\left(\frac{\cos kh - \cancel{i\sin kh} - 2 + \cos kh + \cancel{i\sin kh}}{h^2}\right)$$

$$\cos kh = 1 - 2\sin^2\frac{kh}{2}$$

$$= e^{ikx_j}\left(\frac{2\cos kh - 2}{h^2}\right) = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{-\lambda_k^2} e^{ikx_j}$$

Let $-\lambda_k^2 = \left(-\dfrac{4\sin^2\frac{kh}{2}}{h^2}\right)$. Then: $\tilde{D}\,\overrightarrow{e^{ikx}} = -\lambda_k^2\,\overrightarrow{e^{ikx}}$.