Lecture 4:

Recall: Analytic Spectral method to solve

$$L\,u(x) = g(x).$$

Find basis functions $\{\phi_n(x)\}_{n=1}^{\infty}$ such that:

$$L\,\phi_n(x) = \sum_{j=1}^{N} \lambda_j^n \phi_j(x)$$

$$g(x) \approx \sum_{j=1}^{N} b_j \phi_j(x).$$

Let $u(x) = \sum_{j=1}^{N} a_j \phi_j(x).$

Then: $L\,u(x) = g(x) \Rightarrow \sum_{j=1}^{N} a_j \sum_{k=1}^{N} \lambda_k^j \phi_k(x) = \sum_{j=1}^{N} b_j \phi_j(x)$

Comparing coefficients $\Rightarrow$ Diff. eqt becomes algebraic eqts.

Example: Consider : $u_{tt} = u_{xx}$ where $x \in (0, \pi)$, $t > 0$ such that

$$\begin{cases} u(0,t) = 0 \ , \quad u(\pi, t) = 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

Solution: Assume $u(x,t) = X(x)\,T(t)$.

Consider $L = \frac{\partial^2}{\partial x^2}$. Then, we choose $\{\phi_n(x)\}_{n=1}^{\infty} = \{\sin nx, \cancel{\cos nx}\}_{n=0}^{\infty}$.

$\because u(0,t) = u(\pi, t) = 0 \quad \therefore X(0) = X(\pi)$. We neglect $\cos nx$'s .

We can let $u(x,t) = \sum_{n=1}^{N} a_n(t) \sin nx$.

$$u_{tt} = u_{xx} \Rightarrow \sum_{n=1}^{N} a_n''(t) \sin nx = \sum_{n=1}^{N} (-n^2)\, a_n(t) \sin nx$$

Comparing coefficients $\Rightarrow a_n''(t) = -n^2 a_n(t)$.

$\therefore$ $a_n(t)$ = eigenfunction with eigenvalue $-n^2$.

$\therefore$ $a_n(t) = a_n \cos nt + b_n \sin nt \quad (a_n, b_n \in \mathbb{R})$

$\therefore$ $u(x,t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx$

Now, a_n and b_n can be determined by initial condition:

$$u(x,0) = \phi(x) = \sum_{n=1}^{N} a_n \sin nx \Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} \phi(x) \sin nx\,dx$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{N} n b_n \sin nx \Rightarrow b_n = \frac{2}{n\pi}\int_0^{\pi} \psi(x) \sin nx\,dx$$

(WHY?)

(Check!)

Recall: Many times we need to approximate $f(x)$ by:

$$f(x) = \sum_{k=0}^{N} a_k \cos kx + b_k \sin kx \quad \text{where}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \; ; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \; ; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Real Fourier Series)

Consider $f(x) \in V = \{$ real-valued 2π-periodic smooth functions $\}$.

Then, the real Fourier Series of $f(x)$ is given by:

$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$, where $\{a_k\}$ and $\{b_k\}$ are given by:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \; ; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \; ; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Complex Fourier Series)

Consider $f(x) \in W := \{\text{complex-valued } 2\pi\text{-periodic smooth functions}\}$

Then, the complex Fourier Series is given by:

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{where } \{C_k\} \text{ is determined by:}$$

$$C_k = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-\overset{\sqrt{-1}}{\overset{\shortparallel}{i}}kx}\, dx}_{} \quad \left(\text{Here, } e^{ikx} = \cos kx + i \sin kx\right)$$

The integration is computed separately for the real part and imaginary part.

Question: How well does it approximate $f(x)$?

Consider: $V_N = \left\{ F(x) = \sum_{k=0}^{N} A_k \cos kx + B_k \sin kx : A_k, B_k \in \mathbb{R} \right\}$

For any 2π-periodic function, define:

$$\|f - F\|^2 := E(A_0, A_1, \dots, A_N, B_1, B_2, \dots, B_N)$$

$$:= \int_0^{2\pi} \left(f(x) - \left(\sum_{k=0}^{N} A_k \cos kx + B_k \sin kx \right) \right)^2 dx$$

Remark: $\|f - F\|$ is called the least square error between f and F.

Theorem: $E(a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N) = \min\limits_{\forall A_k, B_k \in \mathbb{R}} E(A_0, \ldots, A_N, B_1, \ldots, B_N)$

where:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \; ; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \; ; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Proof: Assume $A_0, \ldots, A_N, B_1, \ldots, B_N$ are the minimizer of E.

Then: $\frac{\partial E}{\partial A_i} = 0$; $\frac{\partial E}{\partial B_i} = 0$.

$$\frac{\partial E}{\partial A_k} = \frac{\partial}{\partial A_k}\int_0^{2\pi}\left(f(x) - \left(\sum_{j=0}^{N} A_j \cos jx + B_j \sin jx\right)\right)^2 dx$$

$$= -2\int_0^{2\pi}\left(f(x) - \sum_{j=0}^{N} A_j \cos jx + B_j \sin jx\right)\cos kx$$

$$= -2\int_0^{2\pi} f(x)\cos kx\,dx + 2\pi A_k = 0 \Rightarrow A_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

Similarly, $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$ etc ...

Is this the critical point of the minimizer? HW.