Lecture 12:

Useful Theorem for eigenvalues

Gerschgorin Theorem

Consider $\vec{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$ = eigenvector of $A = (a_{ij})$ with eigenvalue λ. Then: $A\vec{e} = \lambda \vec{e}$.

For $1 \leq i \leq n$, $\displaystyle\sum_{j=1}^{n} a_{ij} e_j = \lambda e_i$

$$\Longleftrightarrow \quad a_{ii} e_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} e_j = \lambda e_i$$

$$\Longleftrightarrow \quad (a_{ii} - \lambda) e_i = - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} e_j$$

$$\therefore \quad |a_{ii} - \lambda| \, |e_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \, |e_j|$$

Let ℓ be the index such that $\underset{\neq 0}{|e_\ell|} \geq |e_j|$ for $\forall j$.

Then: $$|a_{\ell\ell} - \lambda|\,\cancel{|e_\ell|} \leq \sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}|\,|e_j| \leq \sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}|\,\cancel{|e_\ell|}$$

$$\therefore \quad |a_{\ell\ell} - \lambda| \leq \sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}| \qquad \therefore \quad \lambda \in \overline{B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}|\right)}$$

Ball of radius $\sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}|$ centered at $a_{\ell\ell}$.

Note: We don't know ℓ unless we know λ and $\vec{e}$.

BUT, we can conclude:

$$\lambda \in \overline{B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{\ell j}|\right)} \subseteq \bigcup_{i=1}^{n} \overline{B_{a_{ii}}\left(\sum_{\substack{j=1 \\ j \neq \ell}}^{n} |a_{ij}|\right)}$$

Example: Let $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

(Eigenvalues: $\lambda_1 = 3.618$, $\lambda_2 = 2.618$, $\lambda_3 = 1.382$, $\lambda_4 = 0.382$)

For $l = 1, 4$, $\overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{4} |a_{lj}|\right)} = \{\lambda : |\lambda - 2| \leq 1\}$

For $l = 2, 3$, $\overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{4} |a_{lj}|\right)} = \{\lambda : |\lambda - 2| \leq 2\}$

$\therefore \bigcup_{i=1}^{4} \overline{B_{a_{ii}}\left(\sum_{\substack{j=1 \\ j \neq i}}^{4} |a_{ij}|\right)} = \{\lambda : |\lambda - 2| \leq 2\}$

= Ball with radius 2 and center at $(2, 0)$

$\because$ A is symmetric, all eigenvalues are real.

$\therefore$ all eigenvalues are between 0 and 4 $\therefore \rho(A) \leq 4$.

Condition for Jacobi / Gauss-Seidel to converge

Definition: A matrix $M = (a_{ij})$ is called strictly diagonally dominant (SDD) if: $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for $i = 1, 2, \ldots, n$

Theorem: If a matrix A is SDD, then A is non-singular.

Proof: All eigenvalues $\lambda \in \bigcup_{l=1}^{n} \overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{n} |a_{lj}|\right)}$

A is SDD iff $|a_{ll}| > \sum_{\substack{j=1 \\ j \neq l}}^{n} |a_{lj}|$ for $l = 1, 2, \ldots, n$.

$\therefore$ Every ball $\overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{n} |a_{lj}|\right)}$ must NOT contain 0.

$\therefore$ No eigenvalue is 0 and so A is non-singular.

Theorem: The Jacobi method converges if A is SDD.

Proof: Note that:
$$x_i^{m+1} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^m + \frac{f_i}{a_{ii}} \qquad (*)$$

for $i = 1, 2, \ldots, n$

Let $\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$. We have:
$$x_i^* = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^* + \frac{f_i}{a_{ii}} \qquad (**)$$

$(*) - (**)$
$$e_i^{m+1} = -\frac{1}{a_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} e_j^m$$

$$\therefore \; |e_i^{m+1}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^m| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \|\vec{e}^{\,m}\|_\infty \leq r \|\vec{e}^{\,m}\|_\infty$$

$$\left(\|\vec{e}^{\,m}\|_\infty = \max_j \{|e_j^m|\}\right)$$

where $r = \max_i \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \right\} < 1$ for all i.

$$\therefore \quad \|\vec{e}^{\,m+1}\|_\infty \leq \overset{<1}{r}\,\|\vec{e}^{\,m}\|_\infty$$

$$\therefore \quad \|\vec{e}^{\,m}\|_\infty \leq r^m \|\vec{e}^{\,0}\|_\infty$$

$$\therefore \quad \|\vec{e}^{\,m}\|_\infty \to 0 \quad \text{as} \quad m \to \infty$$

Jacobi method converges.

Theorem: The Gauss-Seidel method converges to the solution of $A\vec{x} = \vec{f}$ if A is SDD.

Proof: Gauss-Seidel method can be written as:

$$x_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{m} + \frac{f_i}{a_{ii}} \quad \text{———} \quad (\ast)$$

Let $\vec{x}^{*}$ = sol of $A\vec{x} = \vec{f}$. Then:

Here, $\vec{x}^m = \begin{pmatrix} x_1^m \\ x_2^m \\ \vdots \\ x_n^m \end{pmatrix}$

$$x_i^{*} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{*} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{*} + \frac{f_i}{a_{ii}} \quad \text{——} \quad (\ast\ast)$$

$(\ast) - (\ast\ast)$:

$$e_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{m}$$

Here, $\vec{e}^m = \begin{pmatrix} e_1^m \\ e_2^m \\ \vdots \\ e_n^m \end{pmatrix}$

We will prove that:

$$\|\vec{e}^{m+1}\|_\infty \le r \|\vec{e}^{m}\|_\infty \quad \text{where } r = \max_i \left\{ \sum_{\substack{j=1 \\ j \ne i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \right\}$$

< 1

Induction on i:

When $i=1$, $$|e_1^{m+1}| \le \sum_{j=2}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^m| \le \|\vec{e}^{\,m}\|_\infty \sum_{j=2}^{n} \left|\frac{a_{1j}}{a_{11}}\right|$$
$$\le r\,\|\vec{e}^{\,m}\|_\infty .$$

$\therefore$ The statement is true when $i=1$.

Assume $|e_k^{m+1}| \le r\,\|\vec{e}^{\,m}\|_\infty$ for $k=1,2,\dots,i-1$.

Then: $$|e_i^{m+1}| \le \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^{m+1}| + \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^m|$$
$$\le r\,\|\vec{e}^{\,m}\|_\infty \sum_{j=1}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| + \|\vec{e}^{\,m}\|_\infty \sum_{j=i+1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$\le \|\vec{e}^{\,m}\|_\infty$

$$< \|\vec{e}^{\,m}\|_\infty \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \le r\,\|\vec{e}^{\,m}\|_\infty$$

By M.I., $|e_i^{m+1}| \le r \|\vec{e}^{\,m}\|_\infty$ for $i = 1, 2, \ldots, n$

$\therefore \|\vec{e}^{\,m+1}\|_\infty \le r \|\vec{e}^{\,m}\|_\infty$

$\therefore \|\vec{e}^{\,m}\|_\infty \le r^m \|\vec{e}^{\,0}\|_\infty \to 0$ as $m \to \infty$ as $r < 1$.

Example: Consider $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \vec{b}$.

A is SDD. $\therefore$ Both Jacobi and Gauss-Seidel method converge.

For Jacobi method, $\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Let $M_J = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ -1/10 & 0 \end{pmatrix}$

Eigenvalues of M_J are $\lambda_1 = 1/10$ and $\lambda_2 = -1/10$. $\therefore$ M_J = diagonalizable

$\therefore \rho(M_J) = 1/10$. $\therefore \|\vec{e}^{\,m}\|_\infty \le \left(\frac{1}{10}\right)^m K_J$ constant depending on the initial error $\vec{e}_0$

Recall:

$\vec{e}^{\,m} = \lambda_1^m \left\{ a_1\vec{u}_1 + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^m \vec{u}_i \right\}$ where $\vec{e}^{\,0} = \sum_{i=1}^{n} a_i \vec{u}_i$.

For Gauss-Seidel method,

$$\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

Let $M_{G-S} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ 0 & 1/100 \end{pmatrix}$.

Eigenvalues of M_{G-S} are $\lambda_1 = 1/100$ and $\lambda_2 = 0$. $\therefore$ M_{G-S} = diagonalizable.

$\therefore$ $\rho(M_{G-S}) = 1/100$. $\therefore$ $\|\vec{e}^{\,m}\|_\infty \le \left(\frac{1}{100}\right)^m K_{G-S}$ ← constant depending on initial error.

$\therefore$ G-S converges faster.

<u>Remark</u>: To reduce the error by a factor of 10^{-m}, we need about $k \ge \dfrac{m}{-\log_{10} \rho(M)}$ iterations

Jacobi needs $k \ge \dfrac{m}{-\log_{10}(1/10)} = m$ and G-S needs $k \ge \dfrac{m}{-\log_{10}(1/100)} = m/2$ iterations.

$\therefore$ G-S converges twice as fast as Jacobi.