MATH3310 Computational and Applied Mathematics

Midterm Examination

Please show all your steps, unless otherwise stated. Answer all **seven** questions. The total score is **130**. Your submitted solution will be checked carefully to avoid plagiarism. Discussions amongst classmates are strictly prohibited.

1. (**20pts**) Using integrating factor, solve the following differential equations:

   (a) $y' + \dfrac{2x+3}{x^2+3x}y = \dfrac{2}{x} + 2 - \dfrac{6}{x+3},\ y(1) = 2.$

   (b) Find the general solutions of $-3y'' + 2y = 2x^3 - 16x$.

   (c) Consider the 2nd order ordinary differential equation $x^2y''+3xy'+y = 4\log x,\ x > 0$. Suppose the solutions to the ODE can be expressed as $y(x) = \dfrac{u(x)}{x}$ for some function $u(x)$. Find a general formula for such kind of solutions.

2. (**20pts**) Using spectral method, find a solution $y(x,t)$ to the following partial differential equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2},\ 0 < x < L, t > 0 \tag{1}$$

with the boundary condition

$$y(0,t) = y(L,t) = 0,\ t \geq 0 \tag{2}$$

and the initial conditions

$$y(x,0) = f(x),\ \frac{\partial y}{\partial t}(x,0) = 0,\ 0 \leq x \leq L \tag{3}$$

where $f(x) = \begin{cases} 2hx/L & \text{if } 0 \leq x \leq L/2 \\ 2h(L-x)/L & \text{if } L/2 \leq x \leq L \end{cases}$ and $c,\ L,\ h$ are some positive constants.

3. (**20pts**) Use Fourier transform to find a solution $u(x,t)$ for the following partial differential equation:

$$a\frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} + cu = 0,\ x \in \mathbb{R}, t > 0 \tag{4}$$

given the boundary condition $u(x,0) = g(x)$, where $a,b > 0$, and $c \in \mathbb{R}$. Express your answer in term of convolution. Please explain your answers in details.

(**Hint:** You may need to use the fact that the Fourier transform of $e^{-\alpha x^2}$ is $\sqrt{\frac{\pi}{\alpha}}e^{-\frac{k^2}{4\alpha}}$)

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4. (**20pts**) Consider the differential equation:

$$(*) \qquad a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume u and f are periodically extended to $\mathbb{R}$. Divide the interval $[0, 2\pi]$ into n equal portions, where $n = 2^l$ for some $l > 10$. Let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n \times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j = \frac{u(x_{j+1}) - u(x_{j-1})}{2h} \quad \text{and} \quad (\mathcal{D}_2\mathbf{u})_j = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2}.$$

for $j = 0, 1, 2, ..., n-1$ and $h = \frac{2\pi}{n}$.

(a) Using Taylor expansion, explain why the differential equation (*) can be discretized as:

$$(**) \quad a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}.$$

In other words, explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Find the null spaces of $\mathcal{D}_1$ and $\mathcal{D}_2$. Express your answers in term of the spanning set of the discrete exponential functions $\overrightarrow{e^{ikx}}$'s. Please explain your answers with details.

(c) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in \mathbb{C}$. If $\mathbf{u}$ satisfies (**), show that

$$(a\lambda_k + b\widetilde{\lambda}_k)\hat{u}_k = \hat{f}_k \text{ for some } \lambda_k \text{ and } \widetilde{\lambda}_k,$$

for $k = 0, 1, 2, ..., n-1$. What are λ_k and $\widetilde{\lambda}_k$? Please explain your answer with details.

(d) Let $\mathbf{u}^*$ be one of the solutions of (**). What is the general solution of (**)? Please show and explain your answer with details.

5. (**15pts**) Consider the following iterative scheme:

$$(***) \quad \mathbf{x}^{k+1} = G\mathbf{x}^k + \mathbf{b},$$

where $G \in M_{n\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$. Suppose G admits the following matrix decomposition:

$$G = QJQ^{-1} \text{ where } Q \in M_{n\times n}(\mathbb{R}),\ J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix} \text{ is a block-diagonal matrix,}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} \in M_{n_i\times n_i}(\mathbb{R}) \text{ and } \sum_{k=1}^{m} n_i = n. \text{ Assume } I - G \text{ is invertible.}$$

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Prove that the above iterative scheme (***) converges within n iterations for any initialization if and only if $\lambda_i = 0$ for all $1 \le i \le m$.

6. (**20pts**) We consider a linear system $A\mathbf{x} = \mathbf{f}$, where $A = (a_{ij})_{1\le i,j\le n}$ is a $n \times n$ tri-diagonal matrix (i.e. $a_{ij} = 0$ if $|i-j| \ge 2$). Let $A = L + D + U$, where L, D and U refer to the strictly lower triangular, diagonal and strictly upper triangular parts of A respectively. We consider the following iterative scheme to solve the linear system:

$$(****) \quad (L + 2D)\mathbf{x}^{k+1} = (D - U)\mathbf{x}^k + \mathbf{f},$$

where $k = 0, 1, 2, ...$

   (a) Let C be a tridiagonal matrix. Prove that $\det(C) = \det(C_0 + \frac{1}{\mu}C_- + \mu C_+)$ for any non-zero μ and $n \times n$ matrix C, where $C = C_- + C_0 + C_+$ and C_-, C_0 and C_+ refer to the strictly lower triangular, diagonal and strictly upper triangular parts of C respectively. (**Hint:** You may need to find the relationship amongst C, $C_0 + \frac{1}{\mu}C_- + \mu C_+$ and $Q = \begin{pmatrix} \mu & & & \\ & \mu^2 & & \\ & & \ddots & \\ & & & \mu^n \end{pmatrix}$.)

   (b) Let $\mathcal{L} = (L+2D)^{-1}(D-U)$ and $\mathcal{M} = -D^{-1}(L+U)$. Denote the characteristic polynomials of $\mathcal{L}$ and $\mathcal{M}$ by $P_{\mathcal{L}}$ and $P_{\mathcal{M}}$ respectively. Using (a), prove that:

$$P_{\mathcal{L}}(\mu^2) = \left(\frac{\mu}{2}\right)^n P_{\mathcal{M}}\left(\frac{2\mu^2 - 1}{\mu}\right) \text{ for any nonzero } \mu.$$

   (c) Suppose the eigenvalues of $\mathcal{M}$ are real and positive. Using (b), write the spectral radius of $\mathcal{L}$ in terms of the spectral radius of $\mathcal{M}$. Hence, deduce that the above iterative scheme (****) converges if the Jacobi method to solve (****) converges.

7. (**15pts**) (**Challenging**, you may refer to Lecture 1 to answer this question.) In a practical problem, we are required to find a function $u : [0,1] \times [0,1] \to \mathbb{R}$ with $u(0,y) = 0, u(1,y) = 1, u(x,0) = u(x,1) = x$ that minimizes:

$$E(u) = \int_0^1 \int_0^1 \left[\alpha(x,y)\left(\frac{\partial u}{\partial x}\right)^2 + 2\beta(x,y)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + \gamma(x,y)\left(\frac{\partial u}{\partial y}\right)^2 \right] dxdy$$

   (a) Compute $\frac{d}{dt}|_{t=0}E(u + tw)$, where $w(x,0) = w(x,1) = w(0,y) = w(1,y) = 0$.

   (b) Using (a), prove that if u minimizes E, it satisfies:

$$\nabla \cdot \left(\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \nabla u \right) = 0, \text{ where}$$

   $\nabla\cdot$ refers to the divergence and ∇u refers to the gradient of u. In other words,

$$\nabla \cdot \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} \text{ and } \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}.$$

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