1.

$$\hat{f}(k) = \int_{-1}^{1} x^2 e^{-ikx} \mathrm{d}x = \frac{2\sin k}{k} + \frac{4\cos k}{k^2} - \frac{4\sin k}{k^3}$$

2. By standard derivation, we have the general solution

$$u(x,t) = F(x-t) + G(x+t)$$

with the additional condition $u_t(x,0) = 0$ (since we are only finding one particular solution), where $F(x) = G(x) = \frac{u(x,0)}{2}$.
If you are curious about the deduction, you may refer to
`https://www.math.ubc.ca/~feldman/m267/pdeft.pdf`
So one particular solution to the original PDE is

$$u(x,t) = e^{-|x-t|} + e^{-|x+t|}$$

3. By direct substitution, for fixed j

$$\begin{aligned}\sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}} &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{t=0}^{n-1} f_t e^{-i\frac{2tk\pi}{n}} e^{i\frac{2jk\pi}{n}} \\ &= \frac{1}{n}\sum_{t=0}^{n-1} f_t \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}(j-t)} \\ &= f_j\end{aligned}$$

4.

$$\begin{aligned}\widehat{(f*g)}(k) &= \frac{1}{n}\sum_{j=0}^{n-1}(\sum_{k=0}^{n-1} f_k g_{j-k}) e^{-i\cdot\frac{2jk\pi}{n}} \\ &= n\cdot\frac{1}{n}\sum_{j=0}^{n-1} f_j\cdot e^{-i\cdot\frac{2jk\pi}{n}}\cdot\frac{1}{n}\sum_{j=0}^{n-1} g_j\cdot e^{-i\cdot\frac{2jk\pi}{n}} \\ &= n\cdot\hat{f}(k)\hat{g}(k)\end{aligned}$$

5. (a)

$$\begin{aligned}\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{-2\pi j\frac{pm+qn}{N}} &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{m(k-p)+n(l-q)}{N}} \\ &= f(p,q)\end{aligned}$$

(b) Let $u_{r,s}$ be the entry at (r+1)-th row and (s+1)-th column of the matrix U, here $0 \le r,s \le N-1$. Then from $\hat{f} = UfU$ we can easily get

$$u_{r,s} = \frac{1}{\sqrt{N}} e^{2\pi j\cdot\frac{rs}{N}}$$

(c) Let $u^*_{m,n}$ be the entry at (m+1)-th row and (n+1)-th column of the matrix U^*, here $0 \le m, n \le N-1$. Then

$$u^*_{m,n} = \overline{u_{n,m}} = \frac{1}{\sqrt{N}} e^{-2\pi j \cdot \frac{mn}{N}}$$

Then it is easy to verify that $UU^* = U^*U = I$

6. (a) By Taylor's expansion, we get

$$u(x_{j+2}) = u(x_j) + 2hu'(x_j) + o(2h)$$
$$u(x_{j-2}) = u(x_j) - 2hu'(x_j) + o(2h)$$

so we deduce that

$$u'(x_j) = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} + o(1)$$

Then we can say that when we choose n is sufficiently large (or h is sufficiently small), $\mathcal{D}_1\mathbf{u}$ can approximate $\mathbf{u}'$, or $\mathcal{D}_1$ can approximate $\frac{d}{dx}$.

Similarly,

$$u(x_{j+4}) = u(x_j) + 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$
$$u(x_{j-4}) = u(x_j) - 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$

so

$$u''(x_j) = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2} + o(1)$$

Then we can say that when we choose n is sufficiently large (or h is sufficiently small), $\mathcal{D}_2\mathbf{u}$ can approximate $\mathbf{u}''$, or $\mathcal{D}_2$ can approximate $\frac{d^2}{dx^2}$

(b) By the structure of $\mathcal{D}_1\mathbf{u}$, it can be verified that

$$(\mathcal{D}_1\overrightarrow{e^{ikx}})_j = \frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4h}$$

So it suffices to show that

$$\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}}$$

is independent of the index j, and this value is exactly the eigenvalue of $\mathcal{D}_1$ corresponding $\overrightarrow{e^{ikx}}$.

$$\begin{aligned}\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}} &= \frac{e^{ik\cdot(x_j+2h)} - e^{ik\cdot(x_j-2h)}}{4he^{ikx_j}} \\ &= \frac{e^{i\cdot 2kh} - e^{i\cdot(-2kh)}}{4h} \\ &= \frac{i\sin(2kh)}{2h}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_1$ corresponding the eigenvalue $\frac{i\sin(2kh)}{2h}$ for $k = 0, 1, ..., n-1$.
Similarly,

$$\begin{aligned}\frac{e^{ikx_{j+4}} - 2e^{ikx_j} + e^{ikx_{j-4}}}{16h^2 e^{ikx_j}} &= \frac{e^{ik\cdot(x_j+4h)} - 2e^{ik\cdot x_j} + e^{ik\cdot(x_j-4h)}}{16h^2 e^{ikx_j}} \\ &= \frac{e^{i\cdot 4kh} - 2 + e^{i\cdot(-4kh)}}{16h^2} \\ &= \frac{\cos(4kh) - 1}{8h^2}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_2$ corresponding the eigenvalue $(\frac{i\sin(2kh)}{2h})^2 = \frac{\cos(4kh)-1}{8h^2}$ for $k = 0, 1, ..., n-1$.

(c) Since $\overrightarrow{e^{ikx}}$ are the eigenvectors of $\mathcal{D}_1$ corresponding the distinct eigenvalues, we get that they are linearly independent. So the set contains n linearly independent vectors forms a basis.

(d) By (b) we get $\mathcal{D}_1\overrightarrow{e^{ikx}} = \lambda_k\overrightarrow{e^{ikx}}$, $\mathcal{D}_2\overrightarrow{e^{ikx}} = (\lambda_k)^2\overrightarrow{e^{ikx}}$
so

$$\begin{aligned}a\mathcal{D}_2\mathbf{u} + \mathbf{b}\boldsymbol{\mathcal{D}_1}\mathbf{u} &= a\mathcal{D}_2(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) + b\mathcal{D}_1(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) \\ &= a\sum_{k=0}^{n-1}(\lambda_k)^2\hat{u}_k\overrightarrow{e^{ikx}} + b\sum_{k=0}^{n-1}\lambda_k\hat{u}_k\overrightarrow{e^{ikx}} \\ &= \sum_{k=0}^{n-1}(a(\lambda_k)^2 + b\lambda_k)\hat{u}_k\overrightarrow{e^{ikx}} \\ &= \mathbf{f} \\ &= \sum_{k=0}^{n-1}\hat{f}_k\overrightarrow{e^{ikx}}\end{aligned}$$

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ is a basis, comparing the coefficients leads to the result that we want to prove.