MATH3310 HW1 Sketch of Solution

February 23, 2021

1. (a) the integrating factor $= e^{\int \frac{2}{x}} = x^2$

$$\begin{aligned} x^2\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \cdot \frac{2y}{x} &= x^2 \cdot \frac{10x^2+5x+1}{x} \\ \implies x^2 y &= \int (10x^3+5x^2+x)\mathrm{d}x \\ &= \frac{5}{2}x^4 + \frac{5}{3}x^3 + \frac{1}{2}x^2 + C \\ \implies y &= \frac{5}{2}x^2 + \frac{5}{3}x + \frac{1}{2} + \frac{C}{x^2} \end{aligned}$$

Substituting $y(1) = c > 0$, it yields $C = c - \frac{14}{3}$.

(b) Consider the homogeneous solution of

$$-2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3y = 0$$

By standard techniques from ODE (i.e. let the integrating factor to be $\frac{\mathrm{d}y}{\mathrm{d}x}$), we should obtain

$$y = \alpha_1 \exp\left(\sqrt{\frac{3}{2}}x\right) + \alpha_2 \exp\left(-\sqrt{\frac{3}{2}}x\right)$$

For the non-homogeneous solution, we have $y = Ax^2+Bx+C$. Using the given conditions and comparing coefficients, we have

$$\begin{aligned} A &= 5 \\ B &= \frac{38}{3} \\ C &= -\frac{11}{3} \\ \alpha_1 &= -\frac{13\sqrt{\frac{3}{2}} + \frac{38}{3}\exp(-\sqrt{\frac{3}{2}})}{\sqrt{\frac{3}{2}}(\exp(\sqrt{\frac{3}{2}}) + \exp(-\sqrt{\frac{3}{2}}))} \\ \alpha_2 &= \frac{\frac{38}{3}\exp(\sqrt{\frac{3}{2}}) - 13\sqrt{\frac{3}{2}}}{\sqrt{\frac{3}{2}}(\exp(\sqrt{\frac{3}{2}}) + \exp(-\sqrt{\frac{3}{2}}))} \end{aligned}$$

2. Let

$$g(x) = f(x) - (\sum_{j=0}^{N} a_j \cos(jx) + \sum_{j=1}^{N} b_j \sin(jx))$$

We claim that $\int_0^{2\pi} g(x)\sin(kx)dx = 0$ for any $k = 1, 2, .., N$. Argue this by contradiction, assume that $\int_0^{2\pi} g(x)\sin(kx)dx = A \neq 0$ for some k, then we define that

$$h(x) = g(x) - \frac{A}{\pi}\sin(kx)$$

So we have

$$\begin{aligned}\int_0^{2\pi} h^2(x)dx &= \int_0^{2\pi} g^2(x)dx + \frac{A^2}{\pi^2}\int_0^{2\pi}\sin^2(kx)dx - \frac{2A}{\pi}\int_0^{2\pi} g(x)\sin(kx)dx \\ &= \int_0^{2\pi} g^2(x)dx - \frac{A^2}{\pi} < \int_0^{2\pi} g^2(x)dx\end{aligned}$$

However, by the construction of a_j, b_j, we know that

$$\int_0^{2\pi} h^2(x)dx \geq \int_0^{2\pi} g^2(x)dx$$

We get the contradiction, so $\int_0^{2\pi} g(x)\sin(kx)dx = 0$, and we get $b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx$ for $k = 1, 2, ..., N$; similarly $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx$ for $k = 1, 2, ..., N$.

3. (a) Let $x' = \frac{\pi x}{L}$, then using integration by substition, you should have the conclusion.

(b)

$$\begin{aligned}b_n &= \frac{1}{3}\int_{-3}^{3}(c_1 x + c_2|x|)\sin\left(\frac{n\pi x}{3}\right)\mathrm{d}x \\ &= \frac{c_1}{3}\cdot(-\frac{3}{n\pi})\int_{-3}^{3} x\mathrm{d}(\cos\frac{n\pi x}{3}) \\ &= -\frac{c_1}{n\pi}\{[x\cos(\frac{n\pi x}{3})]_{-3}^{3} - \int_{-3}^{3}\cos(\frac{n\pi x}{3})dx\} \\ &= -\frac{c_1}{n\pi}[6(-1)^n - \frac{3}{n\pi}\cdot\sin(\frac{n\pi x}{3})|_{-3}^{3}] \\ &= \frac{6c_1}{n\pi}(-1)^{n+1}\end{aligned}$$

$$\begin{aligned}a_0 &= \frac{1}{6}\int_{-3}^{3}(c_1 x + c_2|x|)dx \\ &= \frac{3c_2}{2}\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{3}\int_{-3}^{3}(c_1x + c_2|x|)\cos(\frac{n\pi x}{3})dx \\
&= \frac{2c_2}{3}\cdot\int_0^3 x\cos(\frac{n\pi x}{3})dx \\
&= \frac{2c_2}{3}\cdot\frac{3}{n\pi}\int_0^3 x\mathrm{d}(\sin(\frac{n\pi x}{3})) \\
&= \frac{2c_2}{n\pi}\{[x\sin(\frac{n\pi x}{3})]|_0^3 - \int_0^3 \sin(\frac{n\pi x}{3})dx\} \\
&= \frac{6c_2}{n^2\pi^2}((-1)^n - 1) \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-12}{(n\pi)^2}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

Thus, $f(x) = \frac{3c_2}{2} + \sum_{n=1}^{\infty}\frac{6c_1(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty}\frac{-12}{(2n-1)^2\pi^2}\cos(\frac{(2n-1)\pi x}{3})$

4. By standard derivation, we have the general solution

$$
u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(\frac{-8n^2\pi^2 t}{16}\right)\sin\left(\frac{n\pi x}{4}\right)
$$

If you are curious about the derivation, you may refer to this entry: `https://en.wikipedia.org/wiki/Heat_equation#Solving_the_heat_equation_using_Fourier_series`
By comparing coefficients, it yields

$$
\begin{aligned}
u(x,t) =& 5\exp\left(\frac{-8(8^2)\pi^2 t}{16}\right)\sin(2\pi x) \\
& - 5\exp\left(\frac{-8(20^2)\pi^2 t}{16}\right)\sin(5\pi x) \\
& + 10\exp\left(\frac{-8(32^2)\pi^2 t}{16}\right)\sin(8\pi x)
\end{aligned}
$$

5. Let $u(x,t)$ admits a full Fourier series. Since $u(0,t) = u(2\pi,t) = 0$, thus $u(x,t)$ is found to be only a Fourier Sine series. i.e. $u(x,t) = \sum_{k=1}^{\infty} T_k(t)\sin(kx)$. Note that

$$
u_t - u_{xx} = \sum_{k=1}^{\infty} T_k'(t)\sin(kx) + \sum_{k=1}^{\infty} k^2 T_k(t)\sin(kx) = 2t\sin(nx) + t^2\sin(mx)
$$

By comparing coefficients, we have

$$
\begin{aligned}
T_n'(t) + n^2 T_1(t) &= 2t \\
T_m'(t) + m^2 T_m(t) &= t^2
\end{aligned}
$$

Solving the ODEs, we have

$$T_n(t) = \frac{2t}{n^2} - \frac{2}{n^4} + C_1 e^{-n^2 t}$$
$$T_m(t) = \frac{t^2}{m^2} - \frac{2t}{m^4} + \frac{2}{m^6} + C_2 e^{-m^2 t}$$

Using the initial condition, we have $C_1 = \frac{2}{n^4} + 2$, $C_2 = 1 - \frac{2}{m^6}$. Thus,

$$u(x,t) = [\frac{2t}{n^2} - \frac{2}{n^4} + (\frac{2}{n^4} + 2)e^{-n^2 t}]\sin(nx) + [\frac{t^2}{m^2} - \frac{2t}{m^4} + \frac{2}{m^6} + (1 - \frac{2}{m^6})e^{-m^2 t}]\sin(mx)$$

6. (a)

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} \mathrm{d}x \\ &= \int_{-\infty}^{0} e^{-(ik-a)x} \mathrm{d}x + \int_{0}^{\infty} e^{-(ik+a)x} \mathrm{d}x \\ &= -\frac{1}{ik-a} + \frac{1}{ik+a} \\ &= \frac{2a}{a^2+k^2}\end{aligned}$$

(b) Applying Fourier Transform with respect to x, we have

$$\hat{u}_t = -k^2 \hat{u}$$

Thus we have, by solving this ODE,

$$\hat{u} = C(k) e^{-k^2 t}$$

for some coefficient function $C(k)$. Then apply Fourier Transform to the initial condition and result from (a), we have

$$\hat{u}(k,0) = \frac{2a}{a^2+k^2}$$

So we have $C(k) = \frac{2a}{a^2+k^2}$, hence the solution to the equation after Fourier Transform is

$$\hat{u}(k,t) = \frac{2a}{a^2+k^2} e^{-k^2 t}$$

Applying inverse Fourier Transform, we have

$$u(x,t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-k^2 t + ikx}}{a^2+k^2} \mathrm{d}k$$

Another way to express the solution is by convolution. We can see $\hat{u}$ to be a product of $\frac{2a}{a^2+k^2}$ and $e^{-k^2 t}$. By standard computation, we find the inverse Fourier Transform of $\hat{\phi}(k,t) = e^{-k^2 t}$ to be $\phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$. So we have that

$$\hat{u}(k,t) = \frac{2a}{a^2 + k^2} e^{-k^2 t} = \hat{\varphi}(k)\hat{\phi}(k,t)$$

Using the convolution property of Fourier Transform, we have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \varphi(x-y,t)\phi(y)\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} e^{-a|y|}\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t} - a|y|}\mathrm{d}y \end{aligned}$$

In this way, we don't need to find the Fourier Transform of the initial condition function explicitly, namely $C(k)$ or $\hat{\varphi}(k)$.