4 Analytical methods (Part 1)

4.1 Fourier series/Fourier expansion

In this section, we are going to present one of the most powerful analytical methods – Fourier series. Before doing that, let us first recall some complex notations.

Complex numbers. Let $i = \sqrt{-1}$. For a complex number $z = a + i\,b$, its conjugate is $\bar{z} = a - i\,b$, and its magnitude is

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}\,.$$

Complex functions. $f(x)$ is called a complex function, if we can write it in the form

$$f(x) = f_1(x) + i\,f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are two real functions. The conjugate of $f(x)$ is the complex function $f(x) = f_1(x) - i\,f_2(x)$. We shall often use the relations

$$e^{ix} = \cos x + i\,\sin x\,, \quad \overline{e^{ix}} = e^{-ix} = \cos x - i\,\sin x\,.$$

Definition 4.1 (Orthogonal functions). *Two complex functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $[a, b]$ if the following holds*

$$(f, g) := \int_a^b f(x)\overline{g(x)}dx = 0$$

where $\overline{g(x)}$ is the conjugate of $g(x)$. For example, $\{e^{ikx}\}_{k=1}^{\infty}$ is an orthogonal sequence of functions on $[-\pi, \pi]$ or $[0, 2\pi]$, since

$$(e^{ikx}, e^{ilx}) = \int_0^{2\pi} e^{ikx}\overline{e^{ilx}}dx = 0 \quad \forall\; l \neq k$$

In fact,

$$\int_0^{2\pi} e^{ikx}\overline{e^{ilx}}dx = \int_0^{2\pi} e^{ikx}e^{-ilx}dx = \frac{1}{i(k-l)}e^{i(k-l)x}\Big|_0^{2\pi} = 0\,.$$

Similarly one can verify that the following three sequences

$$\{\cos kx\}_{k=0}^{\infty}, \quad \{\sin kx\}_{k=0}^{\infty}, \quad \{\cos kx, \sin kx\}_{k=0}^{\infty}$$

are all orthogonal sequences of functions on $[-\pi, \pi]$ or $[0, 2\pi]$.

Definition 4.2 (Periodic functions). *A function $f(x)$ is called a periodic function with period d if*

$$f(x+d) = f(x) \quad \forall x \ .$$

For example, e^{ikx}, $\cos kx$ and $\sin kx$ are all periodic functions with period 2π. But

$$\cos 2kx, \quad \sin 2kx$$

are periodic functions with period 2π and also π.

Now, let us discuss the *Fourier series*. The idea of the Fourier series is to expand a given function $f(x)$ (maybe discontinuous) in terms of the cosine and sine functions. We will consider the following two types of expansions:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = c_0 + c_{-1}e^{-ix} + c_1 e^{ix} + c_{-2}e^{-2ix} + c_2 e^{2ix} + \cdots \tag{4.1}$$

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \tag{4.2}$$

Note that the right hand sides of (4.1) and (4.2) are all functions with period of 2π. So function $f(x)$ must be also a function with period 2π. Because of the periodicity, we can consider any interval of length 2π for the Fourier expansions (4.1) and (4.2). We often take $[-\pi, \pi]$ or $[0, 2\pi]$. In our subsequent discussions, we will always use the interval $[-\pi, \pi]$.

Now suppose $f(x)$ is a function with period 2π, i.e., $f(x+2\pi) = f(x)$ $\forall x$. In this case, the graph of $f(x)$ in any interval of length 2π will be repeated in its neighboring interval of length 2π.

Real Fourier series

We first discuss how to find the Fourier series (4.2).

We need to find all the coefficients $\{a_k\}$ and $\{b_k\}$. Recall that $\{\cos kx, \sin kx\}$ are orthogonal on $[-\pi, \pi]$, namely for any $k \neq l$,

$$\int_{-\pi}^{\pi} \cos kx \cos lx dx = 0, \tag{4.3}$$

$$\int_{-\pi}^{\pi} \cos kx \sin lx dx = 0, \tag{4.4}$$

$$\int_{-\pi}^{\pi} \sin kx \sin lx dx = 0 \ . \tag{4.5}$$

Find the coefficient a_k in (4.2). Multiply both sides of (4.2) by $\cos kx$, integrate then over $[-\pi, \pi]$ and use the orthogonality (4.3)-(4.5). We have

$$\int_{-\pi}^{\pi} f(x) \cos kx dx = \int_{-\pi}^{\pi} a_k \cos kx \cos kx dx.$$

From this we obtain

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \tag{4.6}$$

since

$$\int_{-\pi}^{\pi} \cos^2 kx dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2kx}{2} dx = \pi \ .$$

Find the coefficient b_k in (4.2). Multiply both sides of (4.2) by $\sin kx$, integrate then over $[-\pi, \pi]$ and use the orthogonality (4.3)-(4.5). We have

$$\int_{-\pi}^{\pi} f(x) \sin kx dx = \int_{-\pi}^{\pi} b_k \sin kx \sin kx dx.$$

From this we obtain

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \, . \tag{4.7}$$

Find the coefficient a_0 in (4.2). Multiply both sides of (4.2) by the constant 1, then integrate over $[-\pi, \pi]$ to obtain

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} a_0 dx,$$

therefore,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \, , \tag{4.8}$$

that is, the first coefficient a_0 is the average of $f(x)$ on $[-\pi, \pi]$.

In summary, we can expand $f(x)$ as follows:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

where all the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \, , \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \, , \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \, . \end{aligned}$$

Complex Fourier series

Now we shall discuss how to find the complex Fourier series (4.1), namely,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \ . \tag{4.9}$$

We need to find all the coefficients $\{c_k\}$. By Definition 3.1, we know that $\{e^{ikx}\}$ are orthogonal on $[-\pi, \pi]$, namely for any $k \neq l$,

$$(e^{ikx}, e^{ilx}) = \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} dx = 0 \quad \forall\, k \neq l.$$

Thus multiply both sides of (4.9) by e^{-ikx} and use the orthogonality of $\{e^{ikx}\}$, we obtain

$$\int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \int_{-\pi}^{\pi} c_k e^{ikx} e^{-ikx} dx \ ,$$

or

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \ . \tag{4.10}$$

That is, the Fourier series is

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots \tag{4.11}$$

with coefficients c_k defined by (4.10).

Remark 4.1. *Note that in the Fourier series (4.11), the function $f(x)$ can be a real function. For a real function, one can choose the real Fourier expansion (4.2) or the complex form (4.11).*

Think about why we can choose the complex form (4.11) for a real function. Any contradiction ?

4.2 Relation between the real and complex Fourier series

There are close relations between the real and complex Fourier series.

(a) The coefficients c_k in the complex form (4.11) can be derived from the coefficients a_k and b_k in the real form (4.2). In fact, we know

$$e^{ikx} = \cos kx + i \sin kx, \quad e^{-ikx} = \cos kx - i \sin kx. \tag{4.12}$$

Multiply both sides of the second equation by $f(x)$ and integrate over $[-\pi, \pi]$. We obtain

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \int_{-\pi}^{\pi} f(x)\cos kx dx - i\int_{-\pi}^{\pi} f(x)\sin kx dx \ ,$$

That implies

$$2c_k = a_k - i\, b_k \ . \tag{4.13}$$

This can be written as

$$c_k = \frac{1}{2}a_k - \frac{i}{2}b_k \ .$$

Similarly, we can derive from the first equation of (4.12):

$$c_{-k} = \frac{1}{2}a_k + \frac{i}{2}b_k \ .$$

(b) The real coefficients a_k and b_k in (4.2) can be recovered from the complex coefficients c_k in (4.11). Using the formula

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right), \quad \sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right) \ .$$

Therefore

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = c_k + c_{-k} \ , \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{1}{i}(c_{-k} - c_k). \end{aligned}$$

4.3 Examples of Fourier series

We now give some examples to illustrate the calculations of the Fourier series.

Example 4.1. *Find the Fourier series of $f(x) = \cos^2 x$.*

Solution. By definition, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}dx = \frac{1}{2} \ ,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+\cos 2x)\cos kx dx = \begin{cases} 0 \ , & k \neq 2 \\ \frac{1}{2} \ , & k = 2 \ , \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+\cos 2x)\sin kx dx = 0 \ .$$

Therefore the Fourier series of $f(x)$ is

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x \ .$$

This is a well-known formula. ♯

• Try the Fourier expansions of the functions $\sin^2 x$, $\cos 2x$, $\sin x + \cos x$, $\cdots$.

Example 4.2. *Find the Fourier series of $f(x) = \delta(x)$ on $[-\pi, \pi]$. This function is called a delta function and it is one of the most important functions used in physics and engineering. The delta function has the following properties*

$$\int_{-\pi}^{\pi} g(x)\delta(x)dx = g(0) \quad \forall\ g \in C\ [-\pi, \pi]$$

and

$$\delta(x) = 0 \quad \text{for any} \quad x \neq 0.$$

Solution. By definition, the Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)dx = \frac{1}{2\pi}\ , \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos kx dx = \frac{1}{\pi}\cos 0 = \frac{1}{\pi}\ , \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin kx dx = 0\ , \end{aligned}$$

therefore

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos kx\ ,\ x \in\ [-\pi, \pi] \tag{4.14}$$

In the complex case,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\ ,$$

so we have

$$\delta(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}\ ,\ x \in\ [-\pi, \pi]\,. \tag{4.15}$$

We have from (4.15) that

$$\begin{aligned} \delta(x) &= \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\left(e^{ikx} + e^{-ikx}\right) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos kx\ , \end{aligned}$$

this is the same as (4.14).

Can the series (4.15) really reflect the behavior of $\delta(x)$?

Let us consider the partial sum of (4.15):

$$\begin{aligned} P_N(x) &= \sum_{k=-N}^{N} e^{ikx} = \frac{1}{2\pi} e^{-iNx} \sum_{k=-N}^{N} e^{i(N+k)x} \\ &= e^{-iNx} \sum_{k=0}^{2N} e^{ikx} \\ &= e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\ &= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} \\ &= \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x} \ . \end{aligned}$$

Then study the following questions

1. For each given N, show that

$$\lim_{x\to 0} P_N(x) = 2(N + \frac{1}{2}).$$

   So $P_N(x)$ will tend to infinity at $x = 0$ when N goes larger and larger.

2. Plot the figure for $P_N(x)$ using Matlab; and calculate the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_N(x) f(x) dx$$

   approximately for $N = 10, 20, 30, 40, 50, 100$. Observe if $P_N(x)$ satisfies that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_N(x) f(x) dx = f(0).$$

   if so, $\frac{1}{2\pi} P_N(x)$ approximates $\delta(x)$.

Odd and even functions. A function $f(x)$ is called an *even function* if it satisfies

$$f(-x) = f(x), \quad \forall\, x\,.$$

And it is called an *odd function* if it satisfies

$$f(-x) = -f(x), \quad \forall\, x\,.$$

It is easy to check the following properties:

For any odd function $f(x)$ on $[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} f(x)dx = 0\,.$$

For any even function $f(x)$ on $[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx\ .$$

Example 4.3. *Find the Fourier series of the odd function*

$$f(x) = x\ ,\ x \in\ [-\pi, \pi]\ .$$

Solution. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \quad \text{(why ?)}$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = 0 \quad \text{(why ?)}$$

Finally for the coefficients b_k, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin kx dx$$

By integration by parts, we obtain

$$b_k = \frac{2}{k\pi}\int_{0}^{\pi}\cos kx dx - \frac{2}{\pi k}x\cos kx\Big|_0^{\pi} = -\frac{2\cos k\pi}{k}\ ,$$

that is,

$$b_1 = 2, \quad b_2 = -\frac{2}{2}, \quad b_3 = \frac{2}{3}, \cdots, \quad b_k = (-1)^{k+1}\frac{2}{k}\ ,$$

so the required Fourier series is

$$x = b_1\sin x + b_2\sin 2x + \cdots = 2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots), \quad -\pi < x < \pi\ .$$

Remark 4.2. *Note that the Fourier series above does not converge at $x = -\pi, \pi$, as the series is 0 at $x = -\pi$ and π.*

4.4 Sine series and cosine series

Every function $f(x)$ can be written as a sum of an even and an odd function, i.e.,

$$f(x) = f_e(x) + f_o(x),$$

with

$$f_e(x) = \frac{f(x) + f(-x)}{2} \ , \quad f_o(x) = \frac{f(x) - f(-x)}{2} \ .$$

A very important observation:

The Fourier series of an even function has only cosine terms, since

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = 0 \ .$$

The Fourier series of an odd function has only sine terms, since

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = 0 \ .$$

Example 4.4. *The function $f(x) = 1$ is known on the half-period $0 < x < \pi$. Find its Fourier series when*

(a) $f(x)$ is extended to $(-\pi, \pi)$ as an even function;

(b) $f(x)$ is extended to $(-\pi, \pi)$ as an odd function.

Solution. By definition of even and odd functions, we have

(a) $f(x)$ is an even function,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} 1 dx = 1 \ ,$$
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{2}{\pi} \int_0^{\pi} \cos kx dx = 0 \ ,$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = 0 \ ,$$

therefore the Fourier series of $f(x)$ is

$$f(x) = 1, \quad -\pi < x < \pi.$$

This recovers the original constant function.

(b) $f(x)$ is an odd function,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \ ,$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = 0 \ ,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{2}{\pi}\int_{0}^{\pi} \sin kx dx$$
$$= -\frac{2}{k\pi}\big((-1)^k - 1\big) = \begin{cases} 0, & k \text{ is even} \\ \frac{4}{k\pi}, & k \text{ is odd} \end{cases} ,$$

so the Fourier series of $f(x)$ is

$$f(x) = \frac{4}{\pi}\Big\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\Big\} \ , \quad -\pi < x < \pi.$$

This is very different from the original constant function.

4.5 Some properties of Fourier series

Introduce

$$E(A_0, A_1, \cdots, A_n, B_0, B_1, \cdots, B_n) = \int_{-\pi}^{\pi} \Big\{f(x) - \sum_{k=0}^{n}(A_k \cos kx + B_k \sin kx)\Big\}^2 dx \, .$$

Then we claim that

The best trigonometric approximation of $f(x)$ on $[-\pi, \pi]$ in the mean-square sense is its Fouries series, i.e.,

$$E(a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_n) = \min_{\forall A_k, B_k \in \mathbb{R}^1} E(A_0, A_1, \cdots, A_n, B_0, B_1, \cdots, B_n)$$

where $\{a_k\}$ and $\{b_k\}$ are the Fourier coefficients of $f(x)$.

To see this, let us assume $\{A_k, B_k\}_{K=0}^{n}$ is a minimizer of E, then

$$\frac{\partial E}{\partial A_k} = 2\int_{-\pi}^{\pi} \Big\{f(x) - \sum_{K=0}^{n}(A_k \cos kx + B_k \sin kx)\Big\} \cos kx dx$$
$$= 2\int_{-\pi}^{\pi} \Big\{f(x) - A_k \cos kx\Big\} \cos kx dx$$
$$= 2\int_{-\pi}^{\pi} f(x) \cos kx dx - 2\pi A_k = 0 \ ,$$

therefore

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx = a_k,$$

for $k \neq 0$. Similarly we have

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = a_0 \ ,$$

and

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = b_k \ , \ \forall k \ .$$

This indicates that the minimizer $\{A_k, B_k\}_{k=0}^{n}$ is the Fourier coefficients of $f(x)$. ♯

• Think about why we can claim what we get is the minimizer, not the maximizer. Think about the difference between $E(\{A_i\},\{B_i\})$ and $E(\{a_i\},\{b_i\})$.

Our second claim is:

Let $F_n(x)$ be the truncated Fourier series

$$F_n(x) = a_0 + a_1\cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx \ ,$$

then we have

$$\int_{-\pi}^{\pi} F_n^2(x)dx \leq \int_{-\pi}^{\pi} f^2(x)dx \,.$$

First we have by using the orthogonality that

$$\begin{aligned}
& \int_{-\pi}^{\pi} \big(f(x) - F_n(x)\big)F_n(x)dx \\
= & \int_{-\pi}^{\pi} f(x)\sum_{k=0}^{n}\big(a_k\cos kx + b_k \sin kx\big)dx - \int_{-\pi}^{\pi}\big\{\sum_{K=0}^{n} a_K \cos kx + b_k \sin kx\big\}^2 dx \\
= & \ 0 \ ,
\end{aligned}$$

thus

$$\begin{aligned}
\int_{-\pi}^{\pi} f^2(x)dx &= \int_{-\pi}^{\pi}\big(f(x) - F_n(x) + F_n(x)\big)^2 dx \\
&= \int_{-\pi}^{\pi}\big(f(x) - F_n(x)\big)^2 dx + 2\int_{-\pi}^{\pi}\big(f(x) - F_n(x)\big)F_n(x)dx + \int_{-\pi}^{\pi} F_n^2(x)dx \\
&= \int_{-\pi}^{\pi}\big(f(x) - F_n(x)\big)^2 dx + \int_{-\pi}^{\pi} F_n^2(x)dx \\
&\geq \int_{-\pi}^{\pi} F_n^2(x)dx \,.
\end{aligned}$$

$\sharp$

- Think about the interesting question. If we define a sequence $\{\alpha_n\}$ by

$$\alpha_n = \int_{-\pi}^{\pi} F_n^2(x)dx,$$

then the sequence $\{\alpha_n\}_{n=0}^{\infty}$ must be monotonely increasing.

4.6 Solution of the Laplace's equation

In this section, we are going to apply the Fourier series to solve an important differential equation, i.e., the Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega \tag{4.16}$$

with the boundary condition

$$u(x, y) = u_0(x, y), \quad (x, y) \in \partial\Omega \tag{4.17}$$

where Ω is the unit circle, i.e.,

$$\Omega = \{(x, y);\ x^2 + y^2 < 1\}.$$

Since Ω is a circle, it is easier to use the polar coordinates:

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\,.$$

Under the transformation, we have

$$u(x, y) = u(r\cos\theta, r\sin\theta) = w(r, \theta)\,.$$

Further, the domain Ω and the equation (4.16) are transformed into

$$\omega = \{(r, \theta);\ 0 \le r < 1,\ \ -\pi \le \theta < \pi\}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\ . \tag{4.18}$$

The boundary condition (4.17) changes into

$$w(1, \theta)\ \ =\ \ u_0(\cos\theta, \sin\theta)\,. \tag{4.19}$$

We are now going to find the solutions of (4.18). First, we can easily check that the following functions

$$1,\ r\cos\theta,\ r\sin\theta,\ r^2\cos 2\theta,\ r^2\sin 2\theta, \cdots \tag{4.20}$$

are all solutions of (4.18). For example, we take $w(r, \theta) = r^k\cos k\theta$ for $k \ge 2$, then

$$w_r = kr^{k-1}\cos k\theta\ ,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rw_r) = k^2 r^{k-2}\cos k\theta\ ,$$

while

$$\frac{1}{r^2} w_{\theta\theta} = -k^2 r^{k-2} \cos k\theta \ ,$$

therefore $w(r, \theta) = r^k \cos k\theta$ is a solution to the equation (4.18). Note that (4.18) is a linear equation, so any combination of two solutions $w_1(r, \theta)$ and $w_2(r, \theta)$ is still a solution (**why ?**). Thus the following combination of the above special solutions is a generenal solution:

$$w(r, \theta) = a_0 + a_1 r \cos\theta + b_1 r \sin\theta + \cdots + a_k r^k \cos k\theta + b_k r^k \sin k\theta + \cdots \ , \qquad (4.21)$$

where a_k and b_k are arbitrary constants.

But we have to determine the coefficients a_k and b_k. This can be done by using the boundary condition (4.19). For this, we let $r = 1$ in (4.21) and obtain

$$w(1, \theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + \cdots + a_k \cos k\theta + b_k \sin k\theta + \cdots \ .$$

We know that $w(1, \theta) = u_0(\cos\theta, \sin\theta)$, so the coefficients a_k and b_k are nothing else but the Fourier coefficients of u_0, i.e.,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(\cos\theta, \sin\theta) d\theta \ , \qquad (4.22)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u_0(\cos\theta, \sin\theta) \cos k\theta d\theta \ , \ k = 1, 2, \cdots \qquad (4.23)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u_0(\cos\theta, \sin\theta) \sin k\theta d\theta \ , \ k = 1, 2, \cdots \ . \qquad (4.24)$$

This indicates that $w(r, \theta)$ in (4.21) is the desired solution of the boundary value problem (4.18) with the coefficients a_k and b_k given by (4.22)-(4.24).

Example 4.5. *Find the solution of the following Laplace equation:*

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & 0 \le r < 1, \quad -\pi \le \theta < \pi \\ u(1, \theta) = \theta, & -\pi \le \theta < \pi \,. \end{cases}$$

and

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & 0 \le r < 1, \quad -\pi \le \theta < \pi \\ u(1, \theta) = \delta(\theta), & -\pi \le \theta < \pi \,. \end{cases}$$

4.7 Orthogonal functions

In this section, we introduce some further knowledge on orthogonal functions.

For a given positive function $w(x)$ on $[a, b]$, we define an inner product

$$(f, g)_\omega = \int_a^b \omega(x)\, f(x)\, g(x) dx$$

for any two real functions $f(x)$ and $g(x)$ on $[a, b]$. And $\omega(x)$ will be called a *weight function*. We will often use the following norm:

$$\|f\|_\omega = \Big\{ \int_a^b \omega(x)\, f^2(x)\, dx \Big\}^{\frac{1}{2}}.$$

Definition 4.3 (Weighted orthogonal functions). *Let $f(x)$ and $g(x)$ be two real functions on $[a, b]$. $f(x)$ is said to be orthogonal to $g(x)$ with respect to the inner product $(\cdot, \cdot)_\omega$ if $(f, g)_\omega = 0$.*

A sequence of functions $\{f_k\}_{k=0}^\infty$ is said to be orthonormal with respect to the inner product $(\cdot, \cdot)_\omega$ if the following holds:

$$(f_m, f_n)_\omega = 0 \quad \forall\, m \neq n$$

and each f_k is unitary, i.e.,

$$\|f_k\|_\omega = 1\,.$$

• Check if function $\cos x$ is orthogonal to $g(x) = \sin x$ with respect to the inner product $(\cdot, \cdot)_\omega$ for $\omega(x) = 1, x, x^2$.

• Verify that any sequence of orthogonal functions $\{g_k\}_{k=1}^\infty$ on the interval $[a, b]$ are linearly independent.

Now we are going to demonstrate that

Any sequence of linearly independent functions $\{\phi_k\}_{k=0}^\infty$ defined on $[a, b]$ can generate a sequence of functions $\{q_k\}_{k=0}^\infty$ which are orthonormal with respect to the inner product $(\cdot, \cdot)_\omega$.

Gram-Schmidt orthogonalization is one of such orthogonalizing techniques. Below we introduce the Gram-Schmidt orthogonalization.

Gram-Schmidt orthogonalization process.

Given a sequence $\{\phi_k\}_{k=0}^{\infty}$ of linearly independent functions defined on $[a,b]$, we are going to construct a sequence of orthonormal functions $\{q_k\}_{k=0}^{\infty}$ as follows:

0) Set
$$\widetilde{q}_0(x) = \phi_0(x)\,.$$
Normalize $\widetilde{q}_0(x)$:
$$q_0(x) = \frac{\widetilde{q}_0(x)}{\|\widetilde{q}_0\|_\omega}\ ;$$

1) Set
$$\widetilde{q}_1(x) = \phi_1(x) - \alpha_{10}\, q_0(x)\,,$$
choose α_{10} such that
$$(\widetilde{q}_1, q_0)_\omega = \int_a^b \omega(x)\, \tilde{q}_1(x) q_0(x) dx = 0,$$
that gives,
$$\alpha_{10} = (\phi_1, q_0)_\omega = \int_a^b \omega(x)\, \phi_1(x) q_0(x) dx.$$
Normalize $\widetilde{q}_1(x)$:
$$q_1(x) = \frac{\widetilde{q}_1(x)}{\|\widetilde{q}_1\|_\omega}.$$

k) Suppose $q_0, q_1, \cdots, q_k$ are constructed such that
$$(q_i, q_j)_\omega = 0 \quad \forall i \neq j \quad \text{ and } \quad \|q_i\|_\omega = 1\ .$$
We then construct q_{k+1} by
$$\widetilde{q}_{k+1}(x) = \phi_{k+1}(x) - \Big\{\alpha_{k+1,0} q_0(x) + \cdots + \alpha_{k+1,k} q_k(x)\Big\}$$
with
$$\alpha_{k+1,i} = (\phi_{k+1}, q_i)_\omega, \quad i = 0, 1, \cdots, k.$$
Normalize q_{k+1}:
$$q_{k+1}(x) = \frac{\widetilde{q}_{k+1}(x)}{\|\widetilde{q}_{k+1}\|_\omega}.$$

Then the sequence $\{q_k\}_{k=0}^{\infty}$ constructed above is an orthonormal sequence, i.e.,
$$(q_i, q_j)_\omega = 0 \quad \forall\ i \neq j\ ; \quad \|q_i\|_\omega = 1\ .$$

Example 4.6. *Given the sequence of polynomials*

$$1, x, x^2, \cdots, x^k, \cdots,$$

on the interval $[-1, 1]$*, use the Gram-Schmidt orthogonalization process to construct an orthonormal sequence of polynomials, and write down the first three constructed polynomials explicitly.*

<u>**Solution**</u> (exercise). The three polynomials are

$$P_0(x) = \frac{1}{\sqrt{2}} \ , \ P_1(x) = \frac{\sqrt{3}}{\sqrt{2}} x \ , \ P_2(x) = \frac{\sqrt{45}}{\sqrt{8}} \Big(x^2 - \frac{1}{3} \Big) \ .$$

Example 4.7. *Check if the Chebyshev polynomials*

$$T_0(x) = 1, \ T_1(x) = x, \ T_2(x) = 2x^2 - 1, \ T_3(x) = 4x^3 - 3x$$

are orthogonal on $[-1, 1]$ *with respect to the weight function* $w(x) = 1/\sqrt{1 - x^2}$.

<u>**Solution**</u>. Use the transformation $x = cos\theta$.

Example 4.8. *Based on a given sequence of functions* $\{\phi_i(x)\}_{k=0}^{\infty}$*, which is orthogonal with respect to the inner product* $(\cdot, \cdot)_\omega$*, use the Gram-Schmidt orthogonalization to construct an orthonormal sequence of functions with respect to* $(\cdot, \cdot)_\omega$*. (*exercise*)*

Example 4.9. *Expand a given function* $f(x)$ *on* $[a, b]$ *in terms of a given orthogonal sequence of functions* $\{\phi_k(x)\}_{k=1}^{\infty}$ *with respect to the inner product* $(\cdot, \cdot)_\omega$*.*

<u>**Solution**</u>. Let

$$f(x) = \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) + \alpha_3 \phi_3(x) + \cdots \ .$$

Think about how to find the coefficients $\{\alpha_k\}$. $\sharp$

4.8 Fourier transform

Fourier transforms play a very important role in mathematics, physics and engineering.

4.8.1 Definition and examples

Recall that

$$e^{ikx} = \cos(kx) + i\,\sin(kx).$$

From this expression, we can easily see that the magnitude of k determines the intensity of the oscillation of function $\exp(ikx)$, and k measures the frequencies of the oscillation.

To better understand the relation between the magnitude of k and the oscillation of $\exp(ikx)$, , one may plot and compare the figures of $\sin \pi x$, $\sin 4\pi x$ and $\sin 8\pi x$.

Definition 4.4. *For a given function $f(x)$ defined on $(-\infty, \infty)$, the Fourier transform of f is a function $\widehat{f}$ depending on* **frequency***:*

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx\,, \quad -\infty < k < \infty\,. \tag{4.25}$$

The inverse Fourier transform of $\widehat{f}(k)$ recovers the original function $f(x)$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx}dk\,, \quad -\infty < x < \infty\,. \tag{4.26}$$

Example 4.10. *Find Fourier transform of the delta function $f(x) = \delta(x)$.*

<u>**Solution**</u>.

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}dx = 1, \quad \text{for all frequencies } k\ .$$

So the Fourier transform of the delta function is a constant function.

Example 4.11. *Find the Fourier transform of the function:*

$$f(x) = \textit{ square pulse } = \begin{cases} 1\,, & |x| \le a \\ 0\,, & |x| > a \end{cases} \quad .$$

<u>**Solution**</u>.

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-a}^{a} e^{-ikx}dx = \frac{2\sin ka}{k}\ .$$

- Think about whether this function $\widehat{f}(k)$ makes sense at $k = 0$.

Example 4.12. *For $a > 0$, find the Fourier transform of the function:*

$$f(x) = \begin{cases} e^{-ax}\,, & x \geq 0 \\ -e^{ax}\,, & x < 0\ . \end{cases}$$

<u>**Solution**</u>. By definition, we have

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_0^{\infty} e^{-ax-ikx}dx + \int_{-\infty}^{0} -e^{ax-ikx}dx \\ &= \frac{1}{a+ik} - \frac{1}{a-ik} = \frac{-2ik}{a^2+k^2}\ . \end{aligned}$$

• Justify the above process yourself.

Example 4.13. *Find the Fourier transform of*

$$f(x) = \textit{sign function} = \begin{cases} 1\,, & x > 0 \\ -1\,, & x < 0\ . \end{cases}$$

<u>**Solution**</u>. We have

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_0^{\infty} e^{-ikx}dx + \int_{-\infty}^{0} -e^{-ikx}dx.$$

But what is $e^{-ikx}\big|_0^{\infty}$? It is difficult to know.

To solve this problem, we consider the function

$$f_a(x) = \begin{cases} e^{-ax}\,, & x > 0 \\ -e^{ax}\,, & x < 0 \end{cases}\,,$$

it is easy to see that

$$\lim_{a\to 0^+} f_a(x) = f(x)\,,$$

then we can compute as follows:

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \lim_{a\to 0^+} f_a(x)e^{-ikx}dx \\ &= \lim_{a\to 0^+} \int_{-\infty}^{\infty} f_a(x)e^{-ikx}dx = \lim_{a\to 0^+} \frac{-2ik}{a^2+k^2} \\ &= \frac{-2i}{k} = \frac{2}{ik}\ . \end{aligned}$$

Example 4.14. *Find the Fourier transformation of the constant function*

$$f(x) = 1, \quad \forall\, x \in (-\infty, \infty)\ .$$

Solution. We have

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} e^{-ikx} dx = \int_{0}^{\infty} e^{-ikx} dx + \int_{-\infty}^{0} e^{-ikx} dx \\ &= \lim_{a\to 0^+} \{ \int_{0}^{\infty} e^{-ax} e^{-ikx} dx + \int_{-\infty}^{0} e^{ax} e^{-ikx} dx \} \\ &= \lim_{a\to 0^+} \{ \frac{1}{a+ik} + \frac{1}{a-ik} \} = \begin{cases} 0 & k \neq 0 \\ ? & k = 0 \end{cases} \end{aligned}$$

What is $\widehat{f}(0)$? Note that $\widehat{f}(k)$ looks like a delta function. Let $\widehat{f}(k) = \alpha\delta(k)$, then by the inverse Fourier transform we have

$$1 = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} dk = \frac{\alpha}{2\pi}\ ,$$

so

$$\alpha = 2\pi$$

or

$$\widehat{f}(k) = 2\pi\delta(k), \quad -\infty < k < \infty\ .$$

4.8.2 Two identities for Fourier transforms

(1) For a function $f(x)$ on $(-\infty, \infty)$ and its Fourier transform $\widehat{f}(k)$, we have

$$2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(k)|^2 dk\,. \tag{4.27}$$

(2) The inner product of any functions f and g satisfies

$$2\pi \int_{-\infty}^{\infty} f(x)\bar{g}(x) dx = \int_{-\infty}^{\infty} \widehat{f}(k)\bar{\widehat{g}}(k) dk$$

where $\bar{g}(x)$ is the conjugate of $g(x)$.

Example 4.15. *Check the relation (4.27) for the following function*

$$f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \ . \end{cases}$$

<u>**Solution**</u>. We have

$$2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_0^{\infty} e^{-2ax} dx = \frac{\pi}{a},$$

while

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_0^{\infty} e^{-ax} e^{-ikx} dx \\ &= -\frac{1}{a+ik} e^{-ax-ikx}\Big|_0^{\infty} = \frac{1}{a+ik} \ , \end{aligned}$$

therefore

$$\int_{-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 dx = \int_{-\infty}^{\infty} \frac{dk}{|a+ik|^2} = \int_{-\infty}^{\infty} \frac{dk}{a^2+k^2} = \frac{\pi}{a} \ ,$$

that verifies (4.27). Here we have used the transformation $k = a\cos\theta/\sin\theta$. ♯

4.8.3 Important properties of Fourier transform

This subsection discusses some more properties of Fourier transforms.

(1) One can directly verify from definition that for any complex number α,

$$\widehat{\alpha f}(k) = \alpha \widehat{f}(k).$$

(2) One can directly verify from definition that

$$\widehat{f+g}(k) = \widehat{f}(k) + \widehat{g}(k).$$

(3) The Fourier transform of $\frac{df}{dx}$ is $ik\widehat{f}(k)$, i.e.,

$$\widehat{\frac{df}{dx}}(k) = ik\widehat{f}(k) \ .$$

To see this, we use

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} dk$$

to obtain

$$\frac{df}{dx}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ik\widehat{f}(k)e^{ikx}dk.$$

Comparing with definition

$$\frac{df}{dx}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\frac{df}{dx}}(k)e^{ikx}dk$$

gives

$$\widehat{\frac{df}{dx}}(k) = ik\widehat{f}(k).$$

(4) The transform of $F(x) = \int_a^x f(x)dx$ is $\frac{\widehat{f}(k)}{ik} + C\,\delta(k)$, i.e.,

$$\widehat{F}(k) = \frac{\widehat{f}(k)}{ik} + C\,\delta(k)$$

To see this, we use

$$\frac{dF(x)}{dx} = f(x),$$

or

$$\frac{d}{dx}\big(F(x)+C\big) = f(x) \quad \forall C \in \mathbb{R}^1.$$

Taking the transform on both sides,

$$ik\big(\widehat{F}(k) + 2C\,\pi\delta(k)\big) = \widehat{f}(k)\ ,$$

this is ,

$$\widehat{F}(k) = \frac{\widehat{f}(k)}{ik} + C\delta(k) \quad \forall\, C \in \mathbb{R}^1\,.$$

(5) The Fourier transform of $F(x) = f(x-d)$ is $e^{-ikd}\widehat{f}(k)$.

In fact, we have

$$\begin{aligned}\widehat{F}(k) =& \int_{-\infty}^{\infty} F(x)e^{-ikx}dx\\ =& \int_{-\infty}^{\infty} f(x-d)e^{-ikx}dx\\ =& \int_{-\infty}^{\infty} f(y)e^{-ik(y+d)}dy\\ =& e^{-ikd}\widehat{f}(k)\ .\end{aligned}$$

(6) The transform of $g(x) = e^{ixd}f(x)$ is $\widehat{f}(k-d)$.

By definition, we have

$$\begin{aligned}\widehat{g}(k) &= \int_{-\infty}^{\infty} g(x)e^{-ikx}dx = \int_{-\infty}^{\infty} f(x)e^{(id-ik)x}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-ik'x}dx = \widehat{f}(k') = \widehat{f}(k-d) \ .\end{aligned}$$

(7) The convolution of G and h is the function

$$u(x) = \int_{-\infty}^{\infty} G(x-y)h(y)dy \ ,$$

we often write

$$u(x) = (G * h)(x)$$

or

$$u(x) = \Big(G * h\Big)(x) = \int_{-\infty}^{\infty} G(x-y)h(y)dy\,.$$

We now show that

$$\widehat{u}(k) = \widehat{G}(k)\,\widehat{h}(k).$$

In fact, we have

$$\begin{aligned}\widehat{u}(k) &= \int_{-\infty}^{\infty} u(x)e^{-ikx}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-y)h(y)e^{-ikx}dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-y)h(y)e^{-ikx}dxdy = \int_{-\infty}^{\infty} h(y)\int_{-\infty}^{\infty} G(x-y)e^{-ikx}dxdy \\ &= \int_{-\infty}^{\infty} h(y)e^{-iky}dy\int_{-\infty}^{\infty} G(x')e^{-ikx'}dx' \quad (\text{let } x-y=x') \\ &= \widehat{G}(k)\,\widehat{h}(k)\,, \quad -\infty < k < \infty\,.\end{aligned}$$

4.8.4 Application of Fourier transform for differential equations

Fourier transforms can be applied to solve different types of differential equations. Here we consider one example.

Consider the differential equation

$$-\frac{d^2u}{dx^2} + a^2u = h(x)\ ,\ -\infty < x < \infty\ . \tag{4.28}$$

In order to solve the equation, we apply the Fourier transform

$$\widehat{u}(k) = \int_{-\infty}^{\infty} u(x)e^{-ikx}\,dx$$

to each term of the equation to obtain

$$-(ik)^2\widehat{u}(k) + a^2\widehat{u}(k) = \widehat{h}(k)\ .$$

This gives

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{a^2+k^2}\ . \tag{4.29}$$

Let $G(x)$ be a function such that

$$\widehat{G}(k) = \frac{1}{a^2+k^2},$$

then we know

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{a^2+k^2} = \widehat{G}(k)\widehat{h}(k).$$

By the convolution property, the solution $u(x)$ can be given by

$$u(x) = (G*h)(x) = \int_{-\infty}^{\infty} G(x-y)h(y)dy\,. \tag{4.30}$$

To find $G(x)$, we consider function $f(x) = e^{-a|x|}$. By definition, we have

$$\begin{aligned}
\widehat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \\
&= \int_{0}^{\infty} e^{-(a+ik)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx \\
&= -\frac{1}{a+ik}e^{-(a+ik)x}\Big|_{x=0}^{x=\infty} + \frac{1}{a-ik}e^{(a-ik)x}\Big|_{x=-\infty}^{x=0} \\
&= \frac{1}{a+ik} + \frac{1}{a-ik} = \frac{2a}{a^2+k^2}\,.
\end{aligned}$$

this shows

$$\widehat{\frac{1}{2a}f}(k) = \frac{1}{a^2+k^2}\,,$$

so we have

$$G(x) = \frac{1}{2a}e^{-a|x|}\,.$$

Now we get from (4.30) that

$$u(x) = \frac{1}{2a}\int_{-\infty}^{\infty} e^{-a|x-y|}h(y)dy, \quad -\infty < x < \infty\,. \tag{4.31}$$

♯

• Check if the function $u(x)$ in (4.31) is indeed a solution to the differental equation (4.28).