Exercise 8

Standard notations are in force. Many problems are taken from [R].

(1)
$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at $t = 0$ and
$$\Phi'(0) = p \int_X |f|^{p-2} f g \, d\mu.$$

Hint: Use the convexity of $t \mapsto |f + tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for $t < 0$.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

(3) Assume, in addition to the hypothesis of the previous problem, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \le \|f\|_s$ if $0 < r < s \le \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{\int_X \log |f| \, d\mu\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

(4) For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

(5) Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

(6) Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h \, d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

(7) Optional. Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t) \, dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $\dfrac{p}{p-1}$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\displaystyle\int F^{p-1}f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].