Exercise 7

For those who have learnt functional analysis, this exercise serves to refresh your memory. For those who have not learnt it, working through the problems gives you some feeling on the subject.

(1) Provide two proofs that $C[0,1]$ is an infinite dimensional vector space.

(2) Show that both $C_c(0,1)$ and $C^1[0,1]$ are not closed subspaces in $C[0,1]$ and hence they are not Banach spaces.

(3) Endow $C[0,1]$ with the norm $\|f\| = \int_0^1 |f(x)|dx$. Determine whether it is complete or not.

(4) Let $C_0(X)$ be the space of all continuous functions vanishing at infinity where X be a locally compact Hausdorff space under the supernorm. A function is called vanishing at infinity if for each $\varepsilon > 0$, there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Show that $C_0(X)$ is the completion of $C_c(X)$. In other words, $C_0(X)$ is complete and $\overline{C_c(X)} = C_0(X)$.

(5) Show that the space of bounded sequences, ℓ^∞, is not separable. Hint: Consider all sequences of the form $a = (a_1, a_2, \cdots)$ where $a_j \in \{0,1\}$.

(6) Let Λ be a bounded linear functional on the normed space X. Show that its operator norm

$$\begin{aligned} \|\Lambda\|_{op} &= \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\} \\ &= \inf\left\{M : |\Lambda x| \leq M\|x\|,\ \forall x \in X\right\}. \end{aligned}$$

(7) Show that a linear functional in a normed space is bounded if and only if its kernel is closed.

(8) For any normed space $(X, \|\cdot\|)$, prove that $(X', \|\cdot\|_{op})$ forms a Banach space.

(9) Consider $C[-1,1]$ under the sup-norm. Let

$$X = \left\{ f \in C[-1,1] : \int_{-1}^0 f = \int_0^1 f = 0 \right\},$$

and $g \in C[-1,1]$ satisfy $\int_0^1 g = 1$, $\int_{-1}^0 g = -1$. Establish the followings:

(a) X is a closed subspace of $C[-1,1]$.

(b) $\|g - f\|_\infty > 1$, $\forall f \in X$.

(c) $\mathrm{dist}(g, X) = 1$.

Hint: $\int_0^1 (g-f) = 0$ if and only if $g \equiv f$. This example shows that the projection property does not hold in $(C[-1,1], \|\cdot\|_\infty)$.

(10) Let X be a Hilbert space and X_1 a proper closed subspace. For x_0 lying outside X_1, let $d = \|x_0 - z\|$ where d is the distance from x_0 to X_1. Show that

$$\langle x, z - x_0 \rangle = 0, \quad \forall x \in X_1.$$

Hint: For $x \in X_1$, one has $\frac{d}{dt}\phi(t) = 0$ at $t = 0$ where $\phi(t) = \|z_0 + tx - x_0\|^2$. Why?

(11) Show that the correspondence $\Lambda \mapsto w$ in Theorem 4.9 is norm preserving.

(12) Let Λ_1 and Λ_2 be two bounded linear functionals on the Hilbert space X. Suppose that they have the same kernel. Prove that there exists a nonzero constant c such that $\Lambda_2 = c\Lambda_1$. Use this fact to give a proof of Theorem 4.8.