Exercise 3

(1) Prove the conclusion of Lebesgue's dominated convergence theorem still holds when the condition "$\{f_k\}$ converges to f a.e." is replaced by the condition " $\{f_k\}$ converges to f in measure".

(2) Find an example in each of the following cases.

(a) A sequence which converges in measure but not at every point.

(b) A sequence which converges pointwisely but not in measure.

(c) A sequence which converges in measure but not in L^1.

(3) Let $f_n, n \geq 1$, and f be real-valued measurable functions in a finite measure space. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a subsubsequence that converges to f a.e..

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\widetilde{\mathcal{M}}$ contain all sets E such that there exist $A, B \in \mathcal{M}$, $A \subset E \subset B$, $\mu(B \backslash A) = 0$. Show that $\widetilde{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$ and if we set $\widetilde{\mu}(E) = \mu(A)$, then $(X, \widetilde{\mathcal{M}}, \widetilde{\mu})$ is a complete measure space.

(5) Show that $\widetilde{\mathcal{M}}$ in the previous problem is the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$.

(6) Here we consider an application of Caratheodory's construction. An *algebra* $\mathcal{A}$ on a set X is a subset of $\mathcal{P}_X$ that contains the empty set and is closed under taking complement and finite union. A *premeasure* $\mu : \mathcal{A} \to [0, \infty]$ is a finitely additive function which satisfies: $\mu(\phi) = 0$ and $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$ whenever E_k are disjoint and $\bigcup_{k=1}^{\infty} E_k \in \mathcal{A}$. Show that the premeasure μ can be extended to a measure on the σ-algebra generated by $\mathcal{A}$. Hint: Define the

outer measure

$$\overline{\mu}(E) = \inf\left\{\sum_k \mu(E_k) : E \subset \bigcup_k E_k, E_k \in \mathcal{A}\right\}.$$

This is called Hahn-Kolmogorov theorem.

The following problems are concerned with the Lebesgue measure. Let $R = I_1 \times I_2 \times \cdots \times I_n$, I_j bounded intervals (open, closed or neither), be a rectangle in $\mathbb{R}^n$. More properties of the Lebesgue measure can be found in the Exercise 4.

(7) For a rectangle R in $\mathbb{R}^n$, define its "volume" to be

$$|R| = (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_n - a_n)$$

where b_i, a_i are the right and left endpoints of I_j. Show that

(a) if $R = \bigcup_{k=1}^{N} R_k$ where R_k are almost disjoint (that's, their interiors are pairwise disjoint), then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

(b) If $R \subset \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

(8) Let $\mathcal{R}$ be the collection of all closed cubes in $\mathbb{R}^n$. A closed cube is of the form $I \times \cdots \times I$ where I is a closed, bounded interval.

(a) Show that $(\mathcal{R}, |\cdot|)$ forms a gauge, and thus it determines a complete measure $\mathcal{L}^n$ on $\mathbb{R}^n$ called the *Lebesgue measure.*

(b) $\mathcal{L}^n(R) = |R|$ where R is a cube, closed or open.

(c) For any set E and $x \in \mathbb{R}^n$, $\mathcal{L}^n(E + x) = \mathcal{L}^n(E)$. Thus the Lebesgue measure is translational invariant.

(9) This problem is optional. Use Hahn-Kolmogorov theorem to construct the Lebesgue measure instead of Problems 7 and 8.