Fall 2021 MATH5011 Real Analysis I

Exercise 2

Notations in lecture notes are in use.

(1) Let g be a measurable function in $[0, \infty]$. Show that

$$m(E) = \int_E g \, d\mu$$

defines a measure on $\mathcal{M}$. Moreover,

$$\int_X f \, dm = \int_X fg \, d\mu, \qquad \forall f \text{ measurable in } [0, \infty].$$

(2) Let $\{f_k\}$ be measurable in $[0, \infty]$ and $f_k \downarrow f$ a.e., f measurable and $\displaystyle\int f_1 \, d\mu < \infty$. Show that

$$\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu.$$

What happens if $\displaystyle\int f_1 \, d\mu = \infty$?

(3) Let f be a measurable function. Show that there exists a sequence of simple functions $\{s_j\}$, $|s_1| \leq |s_2| \leq |s_3| \leq \cdots$, and $s_k(x) \to f(x)$, $\forall x \in X$.

(4) Let $\mu(X) < \infty$ and f be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in [a, b], \ \forall E \in \mathcal{M}, \mu(E) > 0$$

for some $[a, b]$. Show that $f(x) \in [a, b]$ a.e.

(5) Let f be Lebsegue integrable on $[a, b]$ which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every c. Show that f is equal to 0 a.e..

(6) Let $f \geq 0$ be integrable and $\displaystyle\int f\, d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n\to\infty} \int n \log\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right) d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0,1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

(7) Let $\mu(X) < \infty$ and $f_k \to f$ uniformly on X and each f_k is bounded. Prove that

$$\lim_{k\to\infty} \int f_k\, d\mu = \int f\, d\mu.$$

Can $\mu(X) < \infty$ be removed?

(8) Give another proof of Borel-Cantelli lemma in Problem 7, Ex.1, by using integration theory. Hint: Study $g(x) = \displaystyle\sum_{j=1}^{\infty} \chi_{A_j}(x)$.

(9) Let f be a Riemann integrable function on $[a, b]$ and extend it to $\mathbb{R}$ by setting it zero outside $[a, b]$.

(a) Show that f is Lebsegue measurable.

(b) Show that the Riemann integral of f is equal to $\int_{\mathbb{R}} f d\mathcal{L}^1$.

(c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on $[a, b]$ and converges pointwisely to some function which is not Riemann integrable.

(10) Let f be integrable in $(X, \mathcal{M}, \mu)$. Show that for each $\varepsilon > 0$, there is some δ such that

$$\int_E |f| < \varepsilon, \qquad \text{whenever } E \in \mathcal{M},\ \mu(E) < \delta\ .$$

This is called the absolute continuity of an integrable function.