## Exercise 10

- (1) Let  $\mathcal{L}^1$  be the Lebesgue measure on (0,1) and  $\mu$  the counting measure on (0,1). Show that  $\mathcal{L}^1 \ll \mu$  but there is no  $h \in L^1(\mu)$  such that  $d\mathcal{L}^1 = h d\mu$ . Why?
- (2) Let  $\mu$  be a measure and  $\lambda$  a signed measure on  $(X,\mathfrak{M})$ . Show that  $\lambda \ll \mu$  if and only if  $\forall \varepsilon > 0$ , there is some  $\delta > 0$  such that  $|\lambda(E)| < \varepsilon$  whenever  $|\mu(E)| < \delta$ ,  $\forall E \in \mathfrak{M}$ .
- (3) Let  $\mu$  be a  $\sigma$ -finite measure and  $\lambda$  a signed measure on  $(X,\mathfrak{M})$  satisfying  $\lambda \ll \mu$ . Show that

$$\int\!f\,d\lambda=\int\!fh\,d\mu,\quad\forall f\in L^1(\lambda),\ fh\in L^1(\mu)$$

where  $h = \frac{d\lambda}{d\mu} \in L^1(\mu)$ .

- (4) Let  $\mu$ ,  $\lambda$  and  $\nu$  be finite measures,  $\mu \gg \lambda \gg \nu$ . Show that  $\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \frac{d\lambda}{d\mu}$ ,  $\mu$  a.e.
- (5) Show that the completion of  $C_c(X)$  under the sup-norm is  $C_0(X)$  where X is a locally compact, Hausdorff space.
- (6) Provide a proof of Proposition 5.8.
- (7) Show that M(X), the space of all signed measures defined on  $(X,\mathfrak{M})$ , forms a Banach space under the norm  $\|\mu\| = |\mu|(X)$ .
- (8) Show that  $M_r(X)$  is a closed subspace in M(X) on  $(X, \mathcal{B})$  where X is a locally compact Hausdorff space. Hence it is a Banach space.