TA's solution[1] to 5011 midterm exam

Q1(a) Please refer to Lecture notes Chapter 1 Section 1.3 for a proof.

(b) Please refer to Exercise 1 solution Question 7 for a proof.

Q2(a) Please refer to Lecture notes Chapter 1 Theorem 1.14.

(b) Please refer to Lecture notes Chapter 2 Theorem 2.12.

(c) Please refer to Exercise 1 solution Question 10 for a proof.

Q3(a) Since $\mu(X) > 0$ and f is a measurable function in X which is positive almost everywhere, $\mu(\{x \in X : f(x) > 0\}) > 0$. Note that $\{x \in X : f(x) > 0\} = \bigcup_{k=1}^{\infty}\{x \in X : f(x) > \frac{1}{k}\}$ and $\{x \in X : f(x) > \frac{1}{k}\} \subseteq \{x \in X : f(x) > \frac{1}{k+1}\}, \forall k \in \mathbb{N}$. By continuity of measure, $0 < \mu(\{x \in X : f(x) > 0\}) = \lim_{k\to\infty} \mu\{x \in X : f(x) > \frac{1}{k}\}$. Hence, there exists $k_0 \in \mathbb{N}$ such that $\mu\{x \in X : f(x) > \frac{1}{k_0}\} > 0$. Take $\rho = \frac{1}{k_0}$. We are done.

(b) Please refer to Exercise 2 solution Question 6 for a proof.

Q4(a) Plainly $\mu([a, b]) \leq \phi([a, b])$. To show the reverse inequality, let $\{I_k = [a_k, b_k]\}_{k=1}^{\infty}$ be a collection of closed and bounded intervals such that $[a, b] \subseteq \bigcup_k I_k$. Our aim is to show

$$\sum_{k=1}^{\infty} \phi(I_k) \geq \phi([a, b]) = g(b) - g(a).$$

Approach 1

Recall that g is a continuous, non-decreasing function on $\mathbb{R}$. By $[a, b] \subseteq \bigcup_k [a_k, b_k]$, we claim that $[g(a), g(b)] \subseteq \bigcup_k [g(a_k), g(b_k)]$, which may be justified as follows. Given $y \in [g(a), g(b)]$, by the intermediate value theorem, there exists $x \in [a, b]$ such that $y = g(x) \in g([a, b]) \subseteq g(\bigcup_k [a_k, b_k]) \subseteq \bigcup_k [g(a_k), g(b_k)]$.

As a result,

$$g(b) - g(a) = \mathcal{L}([g(a), g(b)]) \leq \mathcal{L}(\bigcup_{k=1}^{\infty}[g(a_k), g(b_k)]) \leq \sum_{k=1}^{\infty} \mathcal{L}([g(a_k), g(b_k)]) = \sum_{k=1}^{\infty} \phi(I_k),$$

which was to be demonstrated.

Approach 2

Fix an $\varepsilon > 0$. Since g is continuous and non-decreasing, there exist r_k, s_k such that

$$\begin{cases} -\infty < r_k < a_k \leq b_k < s_k < \infty \\ g(s_k) - g(b_k) < \varepsilon/2^{k+1} \\ g(a_k) - g(r_k) < \varepsilon/2^{k+1}. \end{cases}$$

It follows that we have

$$[a, b] \subseteq \bigcup_k I_k \subseteq \bigcup_k (r_k, s_k) \subseteq \bigcup_k [r_k, s_k],$$

[1] This solution is adapted from the work by former TAs.

and

$$\varepsilon + \sum_{k=1}^{\infty} \phi(I_k) \geq \sum_{k=1}^{\infty} \phi([r_k, s_k]).$$

As $[a, b]$ is compact and covered by $\{(r_k, s_k)\}$, there is a finite sub-covering, say, $\{(r_k, s_k)\}_{k=1}^{N}$. Let $\{C_\ell\}_{\ell \in L}$ be the connected components of the set $\bigcup_{k=1}^{N}[r_k, s_k]$. Since $[a, b]$ a connected subset of $\bigcup_{k=1}^{N}[r_k, s_k]$, it is contained in, say, C_1. Given $1 \leq k \leq N$, as $[r_k, s_k]$ is connected, we have either $[r_k, s_k] \subseteq C_1$ or $[r_k, s_k] \cap C_1 = \emptyset$. Therefore, $C_1 = \bigcup_{k \in K}[r_k, s_k]$, where $K := \{1 \leq k \leq N : [r_k, s_k] \subseteq C_1\}$. Since connected subsets of $\mathbb{R}$ are exactly singletons and intervals, we see that C_1 is a closed interval, which we denote by $[E_{\min}, E_{\max}]$.

Let $E := \{r_k\}_{k \in K} \cup \{s_k\}_{k \in K}$ be the set of all end points given by $[r_k, s_k], k \in K$. Given $e \in E$ with $e \neq E_{\max}$, we use $e^{\uparrow}$ to denote the immediate successor of e in E. i.e. $e^{\uparrow}$ is the smallest element in E which is greater than e. Noting that $E \subseteq [E_{\min}, E_{\max}]$, we have $[e, e^{\uparrow}] \subseteq [E_{\min}, E_{\max}] = \bigcup_{k \in K}[r_k, s_k]$, whence there exists $k \in K$ such that

$$\frac{e + e^{\uparrow}}{2} \in [r_k, s_k].$$

As a result,

$$\begin{cases} e < \dfrac{e + e^{\uparrow}}{2} \leq s_k \Rightarrow e^{\uparrow} \leq s_k \\ e^{\uparrow} > \dfrac{e + e^{\uparrow}}{2} \geq r_k \Rightarrow e \geq r_k. \end{cases}$$

i.e. $[e, e^{\uparrow}] \subseteq [r_k, s_k]$. Consequently, each such $[e, e^{\uparrow}]$ is contained in some $[r_k, s_k]$, whence

$$\begin{aligned} \varepsilon + \sum_{k=1}^{\infty} \phi(I_k) \geq \sum_{k=1}^{\infty} \phi([r_k, s_k]) \geq \sum_{k \in K} \phi([r_k, s_k]) &= \sum_{k \in K} \sum_{\substack{e \in E \setminus E_{\max} \\ [e, e^{\uparrow}] \subseteq [r_k, s_k]}} \phi([e, e^{\uparrow}]) \\ &= \sum_{e \in E \setminus E_{\max}} \phi([e, e^{\uparrow}]) \sum_{\substack{k \in K \\ [e, e^{\uparrow}] \subseteq [r_k, s_k]}} 1 \geq \sum_{e \in E \setminus E_{\max}} \phi([e, e^{\uparrow}]) \\ &= \phi([E_{\min}, E_{\max}]) \geq \phi([a, b]) \quad \text{since } [a, b] \subseteq [E_{\min}, E_{\max}]. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the result follows.

(b) Let $\mathcal{G}$ be the collection of all closed and bounded intervals in $\mathbb{R}$. As $(\mathcal{G}, \phi)$ forms a gauge, μ is an outer measure on $\mathbb{R}$. We shall apply Caratheodory's criterion to show that μ is a Borel measure. So pick two sets $E, F \subseteq \mathbb{R}$ with $\delta_1 := \operatorname{dist}(E, F) > 0$. We want to show that $\mu(E \cup F) = \mu(E) + \mu(F)$. By subadditivity of μ we only need to show that $\mu(E \cup F) \geq \mu(E) + \mu(F)$.

Let $\varepsilon > 0$. By cutting intervals into smaller ones, we see that

$$\mu(E) = \inf \left\{ \sum_k \phi(I_k) : E \subseteq \bigcup_k I_k, I_k \text{ closed and bounded interval with } \operatorname{diam}(I_k) < \delta_1/2 \right\}.$$

Therefore, we can find a countable collection $\mathcal{I}$ of closed intervals such that $E \cup F \subseteq \bigcup_{J \in \mathcal{I}} J$,

$$\mu(E \cup F) + \varepsilon \geq \sum_{J \in \mathcal{I}} \phi(J),$$

and $\mathrm{diam}(J) < \delta_1/2$ for all $J \in \mathcal{I}$. Thus each $J \in \mathcal{I}$ can only intersect at most one of E and F. Let $\mathcal{I}_1 := \{J \in \mathcal{I} : J \cap E \neq \emptyset\}$ and $\mathcal{I}_2 := \{J \in \mathcal{I} : J \cap F \neq \emptyset\}$. We have $E \subseteq \bigcup_{J\in\mathcal{I}_1} J$, $F \subseteq \bigcup_{J\in\mathcal{I}_2} J$, and $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, whence

$$\begin{aligned}\mu(E \cup F) + \varepsilon &\geq \sum_{J\in\mathcal{I}} \phi(J) \\ &\geq \sum_{J\in\mathcal{I}_1} \phi(J) + \sum_{J\in\mathcal{I}_2} \phi(J) \\ &\geq \mu(E) + \mu(F).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, The result follows.

(c) Since g is continuously differentiable, $\int \chi_{[a,b]} g' d\mathcal{L} = \int_a^b g' d\mathcal{L} = g(b) - g(a)$. By (a), $\int \chi_{[a,b]} d\mu = \mu([a,b]) = g(b) - g(a)$. Thus, $\int \chi_{[a,b]} g' d\mathcal{L} = \int \chi_{[a,b]} d\mu$. By linearity, $\int s g' d\mathcal{L} = \int s d\mu$, $\forall$ step function s.
Note that $\forall f \in C_c(\mathbb{R})$,$\exists$ an incresing sequence of step functions $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to f$ pointwisely. By Monotone Convergence Theorem, $\int f g' d\mathcal{L} = \lim_{n\to\infty} \int s_n g' d\mathcal{L} = \lim_{n\to\infty} \int s_n d\mu = \int f d\mu$.

Q5 Plainly μ is a nonnegative function on $\mathcal{M}$ and $\mu(\emptyset) = 0$. Let $\{E_k\}$ be a countable collection of mutually disjoint sets in $\mathcal{M}$. Writing $E := \bigcup_k E_k$, we would like to show that

$$\mu(E) = \sum_k \mu(E_k).$$

On the one hand, given $F_0 \in \mathcal{M}$, we have

$$\begin{aligned}\sum_k \mu(E_k) &= \sum_k \inf\{\mu_1(E_k \setminus F) + \mu_2(E_k \cap F) : F \in \mathcal{M}\} \\ &\leq \sum_k [\mu_1(E_k \setminus F_0) + \mu_2(E_k \cap F_0)] = \mu_1(E \setminus F_0) + \mu_2(E \cap F_0),\end{aligned}$$

whence $\sum_k \mu(E_k) \leq \mu(E)$ by taking inf over $F_0 \in \mathcal{M}$ on the R.H.S.

To get the reverse inequality, let $\varepsilon > 0$ be fixed. For each k, there exists $F_k \in \mathcal{M}$ such that

$$\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k) \leq \mu(E_k) + \frac{\varepsilon}{2^k}$$

Let $F := \bigcup_k (E_k \cap F_k)$. Note that $F \subseteq E$ and $E \setminus F = \bigcup_k (E_k \setminus F_k)$. Hence

$$\begin{aligned}\mu(E) &\leq \mu_1(E \setminus F) + \mu_2(E \cap F) \\ &= \sum_k \mu_1(E_k \setminus F_k) + \sum_k \mu_2(E_k \cap F_k) \\ &= \sum_k [\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k)] \\ &\leq \sum_k \mu(E_k) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof.