MATH1020 General Mathematics **Fall 2019**

Chapter 10: Complex Numbers

10.1 Introduction

In this chapter, we will discuss complex numbers and their properties. One main reason for introducing complex numbers is to solve polynomials. We learned some quadratic polynomials do not have a real root, for examples,

$$x^2 + 1.$$

The number i is introduced to solve this equation.

Definition 1. Define i to be a number such that

$$i^2 = -1.$$

A complex number is a number of the form $z = a + bi$, where $a, b \in \mathbb{R}$.
The real and imaginary part of z is defined to be $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ respectively. z is said to be purely imaginary if $\mathrm{Re}(z) = 0$.

Remark. Real numbers are also complex numbers, They are those with zero imaginary part.

Arithmetic Operations of Complex numbers

Addition, subtraction, multiplication, division of complex numbers are defined.

For complex numbers $a + bi$ and $c + di$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \quad \text{for } c + di \neq 0.$$

These arithmetic operations on complex numbers share properties similar to those on real numbers.

Proposition 1. Let z_1, z_2, z_3 be complex numbers, then

- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z_1 + z_2 = z_2 + z_1$
- $z_1 + 0 = z_1$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z_1 z_2 = z_2 z_1$
- $0 z_1 = 0$
- $1 z_1 = z_1$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Example 1. Express the followings in the form of $a + bi$

a. $[(1 + 2i) - (4 - 3i)] \cdot (-1 - i)$

b. $\dfrac{-1 + 4i}{(2 + i) + (4 - 4i)}$

Solution. a.

$$\begin{aligned}[(1 + 2i) - (4 - 3i)] \cdot (-1 - i) =& (-3 + 5i) \cdot (-1 - i) \\ =& 3 - 5i + 3i - 5i^2 \\ =& 8 - 2i\end{aligned}$$

b.

$$\begin{aligned}\frac{-1 + 4i}{(2 + i) + (4 - 4i)} =& \frac{-1 + 4i}{6 - 3i} \cdot \frac{6 + 3i}{6 + 3i} \\ =& \frac{-6 - 12 - 3i + 24i}{36 + 9} \\ =& -\frac{2}{5} + \frac{7}{15} i\end{aligned}$$

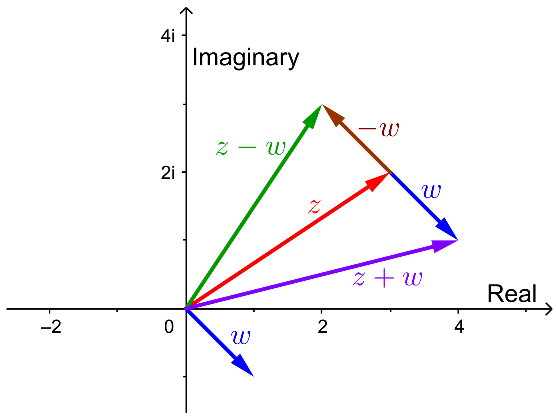
■

10.2 Argand Diagram

A complex number $z = a + bi$ can be represented by the point or vector (a, b) on a plane. The plane is called the complex plane or Argand diagram. Addition and subtraction of complex numbers are similar to those of vectors graphically.

Note that real numbers and purely imaginary numbers are represented by horizontal vectors and vertical vectors respectively. The horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

Example 2. Let $z = 3 + 2i$, $w = 1 - i$



Conjugate, Modulus and Argument

Definition 2. For $z = a + bi$, where $a, b \in \mathbb{R}$, define

Conjugate	$\overline{z} = a - bi$
Modulus	$\lvert z\rvert = \sqrt{a^2 + b^2}$
Argument	$\arg z = \theta$ for $z \neq 0$, where θ is the angle at the origin, measured counterclockwisely from the positive real axis to (a, b) in radian. Note $\arg z$ is only defined up to $2k\pi, k \in \mathbb{Z}$, additively.
Principal argument	$\operatorname{Arg} z = \theta$ if θ is an argument of z with $-\pi < \theta \leq \pi$

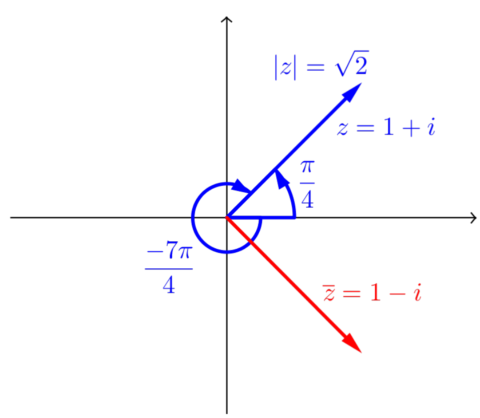
Example 3. For $z = 1 + i$,

$\overline{z} = 1 - i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

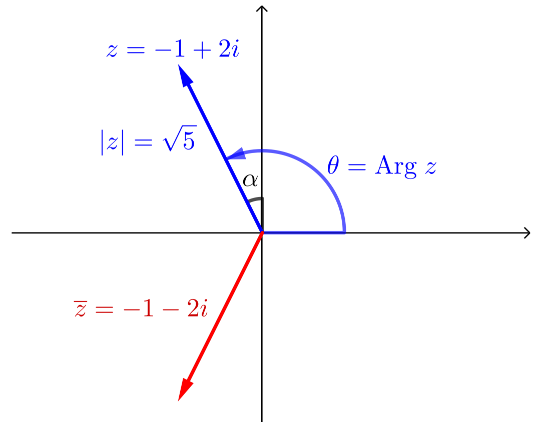
$\arg z = \dfrac{\pi}{4} \ \left(\text{or } \dfrac{\pi}{4} + 2k\pi \text{ for any } k \in \mathbb{Z}. \right)$

$\operatorname{Arg} z = \dfrac{\pi}{4}$



Example 4. For $z = -1 + 2i$,

$$\overline{z} = -1 - 2i$$
$$|z| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$
$$\operatorname{Arg} z = \frac{\pi}{2} + \alpha = \frac{\pi}{2} + \arctan\frac{1}{2}$$
$$\arg z = \frac{\pi}{2} + \arctan\frac{1}{2} + 2k\pi \text{ for any } k \in \mathbb{Z}.$$



Proposition 2. For complex numbers z and w,

- $z + \overline{z} = 2\operatorname{Re} z$
- $z - \overline{z} = 2i\operatorname{Im} z$
- $z\overline{z} = |z|^2$
- $\overline{(\overline{z})} = z$
- $|z| = |\overline{z}|$
- $\overline{z \pm w} = \overline{z} \pm \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$
- $|zw| = |z||w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

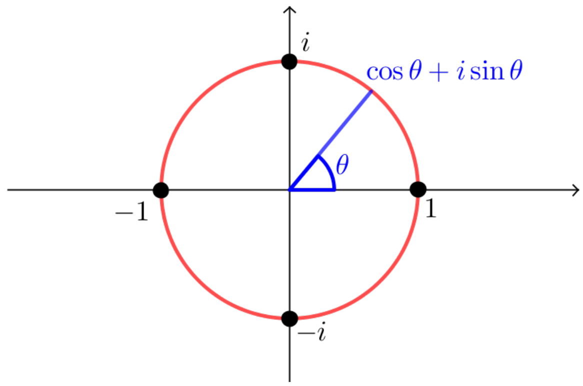
Proof of $z\overline{z} = |z|^2$. Let $z = a + bi$ where $a, b \in \mathbb{R}$. Then

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 - b^2 i^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

□

10.3 Polar form

The complex number $\cos\theta + i\sin\theta$ has modulus 1 and argument θ. Conversely, any complex number with modulus 1 has this form.

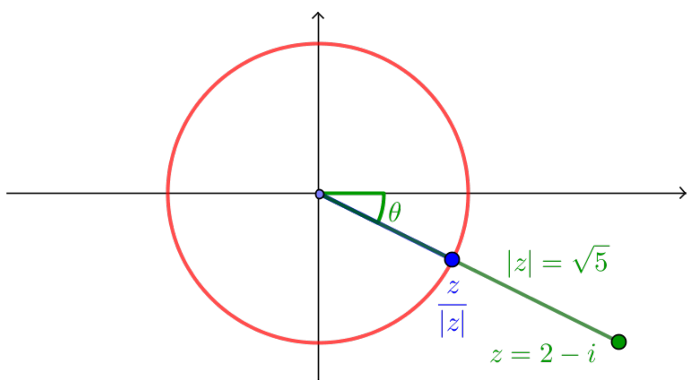


Let $z \neq 0$. Since $|z|$ is a positive real number, $\left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1$ and so

$$\frac{z}{|z|} = \cos\theta + i\sin\theta \tag{*}$$

where $\theta = \arg\frac{z}{|z|} = \arg z$.

Example 5. For $z = 2 - i$,



Rewriting the equation (*) above, z can be expressed as follows, known as the polar form.

Polar Form For $z \neq 0$,

$$z = |z|(\cos\theta + i\sin\theta)$$

where $\arg z = \theta$.

Remark. $z = a + bi$ is called the rectangular form.

Geometrically, $|z|$ and $\theta = \arg z$ determines the length and the direction of the vector representing z on the Argand diagram respectively.

Proposition 3. Let z, w be non-zero complex numbers with polar forms

$$z = |z|(\cos\alpha + i\sin\alpha) \quad \text{and} \quad w = |w|(\cos\beta + i\sin\beta)$$

Then we have the following polar forms:

1. $\overline{z} = |z|[\cos(-\alpha) + i\sin(-\alpha)]$
2. $\dfrac{1}{z} = \dfrac{1}{|z|}[\cos(-\alpha) + i\sin(-\alpha)]$
3. $zw = |z||w|[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$
4. $\dfrac{z}{w} = \dfrac{|z|}{|w|}[\cos(\alpha-\beta) + i\sin(\alpha-\beta)]$

Proof. We will prove properties 2 and 3.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{|z|}\frac{1}{\cos\alpha + i\sin\alpha} \cdot \frac{\cos\alpha - i\sin\alpha}{\cos\alpha - i\sin\alpha} \\
&= \frac{1}{|z|}\frac{\cos\alpha - i\sin\alpha}{(\cos\alpha)^2 - (i\sin\alpha)^2} \\
&= \frac{1}{|z|}\frac{\cos\alpha - i\sin\alpha}{\cos^2\alpha + \sin^2\alpha} \\
&= \frac{1}{|z|}[\cos(-\alpha) + i\sin(-\alpha)]
\end{aligned}$$

$$\begin{aligned}
zw &= |z|(\cos\alpha + i\sin\alpha)|w|(\cos\beta + i\sin\beta) \\
&= |z||w|[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)] \\
&= |z||w|[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]
\end{aligned}$$

□

Example 6. Consider $z = -1 + i$. Then z has modulus and principal argument

$$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \quad \text{and} \quad \theta = \operatorname{Arg} z = \frac{3\pi}{4}.$$

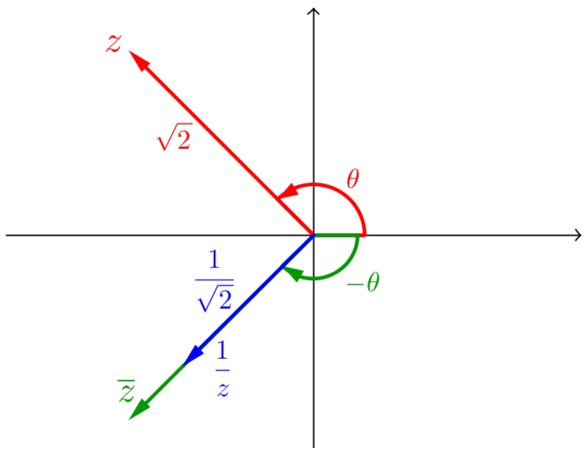
Hence, z has polar form

$$z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Then

$$\overline{z} = |z|[\cos(-\theta) + i\sin(-\theta)] = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$\frac{1}{z} = \frac{1}{|z|}[\cos(-\theta) + i\sin(-\theta)] = \frac{1}{\sqrt{2}}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$



Example 7. Consider $z = 1 + i$ and $w = 2i$. They have arguments $\alpha = \arg z = \frac{\pi}{4}$ and $\beta = \arg w = \frac{\pi}{2}$. The polar forms of z and w are

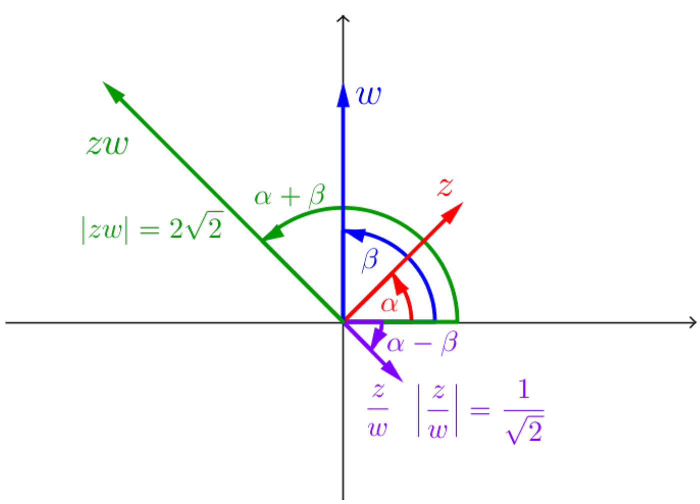
$$z = |z|(\cos\alpha + i\sin\alpha) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w = |w|(\cos\beta + i\sin\beta) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Then

$$zw = |z||w|[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\frac{z}{w} = \frac{|z|}{|w|}[\cos(\alpha-\beta) + i\sin(\alpha-\beta)] = \frac{\sqrt{2}}{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$



Corollary 4. Let z, w be non-zero complex numbers. Then

1. $\arg \overline{z} = \arg \frac{1}{z} = -\arg z$
2. $\arg(zw) = \arg z + \arg w$
3. $\arg \frac{z}{w} = \arg z - \arg w$

10.4 De Moivre's Theorem

In this section, we study powers of complex numbers. The following theorem can be proved using proposition 3.

Theorem 5 (De Moivre's theorem)**.** *For any integer* n*,*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. The theorem is clearly true for $n = 0, 1$. For larger n, for example, $n = 2, 3$, we have

$$\begin{aligned}
(\cos\theta + i\sin\theta)^2 &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) \\
&= \cos(\theta + \theta) + i\sin(\theta + \theta) \\
&= \cos 2\theta + i\sin 2\theta \\
(\cos\theta + i\sin\theta)^3 &= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta) \\
&= (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta) \\
&= \cos(2\theta + \theta) + i\sin(2\theta + \theta) \\
&= \cos 3\theta + i\sin 3\theta
\end{aligned}$$

One can prove the theorem for any positive n inductively.
For negative n, let $m = -n > 0$. By the result above,

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= [(\cos\theta + i\sin\theta)^{-1}]^m \\
&= [\cos(-\theta) + i\sin(-\theta)]^m \\
&= \cos(-m\theta) + i\sin(-m\theta) \\
&= \cos n\theta + i\sin n\theta.
\end{aligned}$$

Hence, the theorem is true of any integer n. □

By using polar form and de Moivre's theorem, powers of complex numbers can be easily computed.

Example 8. Compute $(-1+i)^{10}$.

Solution. In polar form,

$$-1+i=\sqrt{2}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$$

By de Moivre's theorem,

$$\begin{aligned}(-1+i)^{10}&=(\sqrt{2})^{10}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)^{10}\\&=2^5\left[\cos\left(10\cdot\frac{3\pi}{2}\right)+i\sin\left(10\cdot\frac{3\pi}{2}\right)\right]\\&=32(\cos 15\pi+i\sin 15\pi)\\&=32(-1+0i)\\&=-32\end{aligned}$$

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By de Moivre's theorem and binomial theorem, we can express $\sin n\theta$ or $\cos n\theta$ in terms of powers of $\sin\theta$ or $\cos\theta$ and vice versa.

Example 9. Express $\cos 5\theta$ in terms of $\cos\theta$.

Solution. Note that

$$\begin{aligned}\cos 5\theta+i\sin 5\theta=&(\cos\theta+i\sin\theta)^5\\=&\cos^5\theta+5\cos^4\theta(i\sin\theta)+10\cos^3\theta(i\sin\theta)^2\\&+10\cos^2\theta(i\sin\theta)^3+5\cos\theta(i\sin\theta)^4+(i\sin\theta)^5\end{aligned}$$

Comparing the real parts, we have

$$\begin{aligned}\cos 5\theta=&\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\\=&\cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-\cos^2\theta)^2\\=&16\cos^5\theta-20\cos^3\theta+5\cos\theta\end{aligned}$$

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Remark. By comparing the imaginary parts, we have

$$\sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta$$

The following formulas are useful for converting powers of $\sin\theta$ or $\cos\theta$ to sums of multiples of $\sin n\theta$ or $\cos n\theta$.

Let $z = \cos\theta + i\sin\theta$ and n be an integer. By de Moivre's theorem,

$$\begin{aligned} z^n &= \cos n\theta + i\sin n\theta \\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta \end{aligned}$$

Hence,

$$z^n + z^{-n} = 2\cos n\theta \qquad \text{and} \qquad z^n - z^{-n} = 2i\sin n\theta$$

Example 10. Express $\sin^4\theta$ in terms of a sum of multiples of $\sin k\theta$ or $\cos k\theta$.

Solution. Let $z = \cos\theta + i\sin\theta$. Then $z^{-1} = \cos\theta - i\sin\theta$.

$$\begin{aligned} 2i\sin\theta &= z - z^{-1} \\ (2i\sin\theta)^4 &= (z - z^{-1})^4 \\ &= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\ &= (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6 \\ 16\sin^4\theta &= 2\cos 4\theta - 8\cos 2\theta + 6 \\ \sin^4\theta &= \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8} \end{aligned}$$

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10.5 Roots of unity

Let n be a positive integer. A complex number ω is said to be a n-th root of unity if $\omega^n = 1$. For examples, $\pm 1, \pm i$ are the 4-th (fourth) roots of unity.

In general, suppose ω is a n-th root of unity. Then

$$|\omega|^n = |\omega^n| = |1| = 1 \implies |\omega| = 1$$

Let $\omega = \cos\theta + i\sin\theta$. Then

$$\begin{aligned} \omega^n &= (\cos\theta + i\sin\theta)^n \\ 1 &= \cos n\theta + i\sin n\theta \end{aligned}$$

By comparing real and imaginary parts,

$$\cos n\theta = 1 \quad \text{and} \quad \sin n\theta = 0.$$

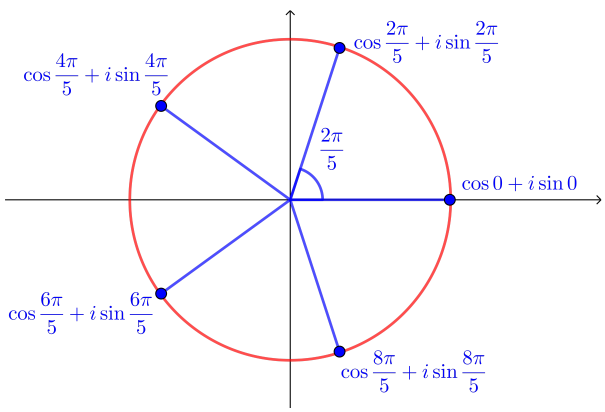
Hence $n\theta = 2k\pi$ and so $\theta = \dfrac{2k\pi}{n}$, where k is an integer. Since sine and cosine have period 2π, we have the following result.

Proposition 6. The solutions of $z^n = 1$, called the n-th roots of unity, are given by

$$\omega = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

Note that the n-th roots of unity are on the unit circle centered at 0 and form the vertices of a regular polygon with n-sides on the Argand diagram.

Example 11. The fifth roots of unity are shown



Proposition 7. If ω is a n-th root of unity and $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

Proof. If ω is a n-th root of unity, then $\omega^n = 1$. Note that

$$(\omega - 1)(\omega^{n-1} + \omega^{n-1} + \cdots + 1) = \omega^n - 1 = 0.$$

Since $\omega - 1 \neq 0$, we conclude that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

□

10.6 Polynomials and Complex roots

Complex numbers are important for solving polynomial equations. Consider a quadratic polynomial $ax^2 + bx + c = 0$ with real coefficients. Recall that if the discriminant $\Delta =$

$b^2 - 4ac < 0$, then the polynomial has non-real and distinct roots. The roots can be computed by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 12. Solve $x^2 + 4x + 7 = 0$.

Solution.

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{-12}}{2} \\ &= \frac{-4 \pm \sqrt{12}i}{2} \\ &= -2 \pm \sqrt{3}i \end{aligned}$$

Alternative Solution. It is also possible to solve the equation by completing the square:

$$\begin{aligned} x^2 + 4x + 7 &= 0 \\ (x^2 + 4x + 4) + 3 &= 0 \\ (x+2)^2 &= -3 \\ x + 2 &= \pm\sqrt{3}i \\ x &= -2 \pm \sqrt{3}i \end{aligned}$$

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The quadratic formula holds even if a, b, c are complex numbers.

Example 13. Solve $(1+i)x^2 - 3x - 1 + i = 0$.

Solution.

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1+i)(-1+i)}}{2(1+i)} \\ &= \frac{3 \pm \sqrt{9+8}}{2(1+i)} \cdot \frac{1-i}{1-i} \\ &= \frac{3 \pm \sqrt{17}}{4}(1-i) \end{aligned}$$

■

Remark. In the last example, $\Delta = 17$ is real and $\Delta > 0$, the roots are distinct but not real. For a quadratic polynomial with non-real coefficients, $\Delta > 0$ does not imply that its roots are real.

There are also formulas for computing roots of cubic (deg 3) and quartic (deg 4) polynomials. However, there is no such formula for polynomials of $\deg \geq 5$ in general. Nevertheless, it is a fact that these polynomials can be factorized into products of linear factors.

Theorem 8 (Fundamental Theorem of Algebra)**.** *Let $f(x)$ be a polynomial with degree ≥ 1. Then $f(x)$ has n complex roots, if counted with multiplicities, and can be factorized as*

$$f(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, c_2, \cdots, c_n$ are the complex roots and a is the leading coefficient.

Example 14. Let $f(x) = 2x^5 + 20x^3 + 50x$. As a degree 5 polynomial, $f(x)$ has 5 complex roots. Note that $f(x)$ can be factorized as

$$\begin{aligned} f(x) &= 2x(x^4 + 10x^2 + 25) \\ &= 2x(x^2 + 5)^2 \\ &= 2x(x + \sqrt{5}i)^2(x - \sqrt{5}i)^2 \end{aligned}$$

Counting with multiplicities, the 5 roots are $0, \sqrt{5}i, \sqrt{5}i, -\sqrt{5}i, -\sqrt{5}i$. The roots $\pm\sqrt{5}i$ are double roots with multiplicity 2.

Real polynomials

A polynomial is called a real polynomial if all its coefficients are real.

Proposition 9. Suppose $f(x)$ is a real polynomial. If c is a root of $f(x)$, then $\overline{c}$ is also a root of $f(x)$.

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Since $f(x)$ is real, $a_i = \overline{a_i}$ for $i = 0, 1, \cdots, n$. Hence,

$$\begin{aligned} f(\overline{c}) &= a_n \overline{c}^{\,n} + a_{n-1}\overline{c}^{\,n-1} + a_1\overline{c} \cdots + a_0 \\ &= \overline{a_n}\;\overline{c}^{\,n} + \overline{a_{n-1}}\;\overline{c}^{\,n-1} + \cdots + \overline{a_1}\;\overline{c} + \overline{a_0} \\ &= \overline{a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0} \\ &= \overline{f(c)} = \overline{0} = 0 \end{aligned}$$

It follows that $\overline{c}$ is a root of $f(x)$. □

Note that if c is not real, then

$$(x - c)(x - \overline{c}) = x^2 - (c + \overline{c})x + c\overline{c} = x^2 - (2\operatorname{Re} c)x + |c|^2$$

is a real polynomial. It is irreducible with discriminant

$$(-2\operatorname{Re} c)^2 - 4(1)|c|^2 = -4(|c|^2 - (\operatorname{Re} c)^2) = -4(\operatorname{Im} c)^2 < 0.$$

From this observation and the fundamental theorem of algebra, one can obtain the following result on the factorization of real polynomials.

Proposition 10. Let $f(x)$ be a real polynomial with degree ≥ 1. Then $f(x)$ can be factorized as

$$f(x) = p_1(x)p_2(x)\cdots p_k(x)$$

where each $p_i(x)$ is a real polynomial, either linear or irreducible quadratic.

Example 15. Factorize $x^4 + 1$ as a product of real linear and/or irreducible quadratic polynomials.

Solution. Using polar form and de Moivre's theorem, the roots of $x^4 + 1$ can be found to be

$$\begin{aligned}
\omega_1 &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\
\omega_2 &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \\
\omega_3 &= \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \overline{\omega_2} \\
\omega_4 &= \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \overline{\omega_1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x^4 + 1 &= (x - \omega_1)(x - \overline{\omega_1})(x - \omega_2)(x - \overline{\omega_2}) \\
&= [x^2 - (\omega_1 + \overline{\omega_1})x + |\omega_1|^2][x^2 - (\omega_2 + \overline{\omega_2})x + |\omega_2|^2] \\
&= (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1),
\end{aligned}$$

where each quadratic factors are irreducible. ∎

10.7 Euler's formula

Recall that for a real number x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The exponential function can also be defined for complex numbers and the same formula still holds.

For any complex number z,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

Put $z = i\theta$ for a real number θ, then

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right)}_{\text{Real part}} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\text{Imaginary pary}} \end{aligned}$$

Note that the real part and imaginary part is the power series of $\cos\theta$ and $\sin\theta$ respectively. Hence

Theorem 11 (Euler's formula)**.** *For a real number θ,*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

As suggested by the notation, $e^{i\theta}$ satisfies the properties of exponential functions. For instance,

- $e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$.
- $\dfrac{e^{i\alpha}}{e^{i\beta}} = e^{i(\alpha-\beta)}$.
- $\left(e^{i\theta}\right)^n = e^{in\theta}$.

These properties follow from proposition 3 and de Moivre's theorem.