MATH2040C Linear Algebra II
2017-18 Solution to Homework 5

Exercise 6.A

2 Taking $x_1 = x_3 = 0, x_2 = 1$ leads to $\|(x_1, x_2, x_3)\| = 0$, so this is not an inner product.

5 If $Tv = 0$, then $Tv = \sqrt{2}v$, then $2\|v\| = \|Tv\| \leq \|v\|$, which implies that $\|v\| = 0$, then $v = 0$. So T is invertible.

12 We prove it by the induction method. When $n = 1$, it obviously holds. Assume for $1 \leq n \leq k$, the inequality holds, then for $n = k + 1$, using the Cauchy-Schwarz Inequality, we have

$$
\begin{aligned}
&(x_1 + \cdots + x_k + x_{k+1})^2 = (x_1 + \cdots + x_k)^2 + x_{k+1}^2 + 2x_{k+1}(x_1 + \cdots + x_k) \\
\leq& k(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + 2x_{k+1}(x_1 + \cdots + x_k) \\
\leq& k(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + (x_{k+1}^2 + x_1^2) + \cdots + (x_{k+1}^2 + x_k^2) = (k+1)(x_1^2 + \cdots + x_k^2 + x_{k+1}^2)
\end{aligned}
$$

14* Firstly the definition of domain of arccos is $[-1, 1]$, where by Cauchy-Schwarz Inequality $\dfrac{\langle x, y\rangle}{\|x\|\|y\|} \in [-1, 1]$. Secondly, since the angel between two vectors is invariant under the scaling of these two vectors, for any nonzero numbers λ_1, λ_2, $\dfrac{\langle \lambda_1 x, \lambda_2 y\rangle}{\|\lambda_1 x\|\|\lambda_2 y\|} = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$. So this definition makes sense.

19

$$
\begin{aligned}
&\|u + v\|^2 - \|u - v\|^2 = \langle u + v, u + v\rangle - \langle u - v, u - v\rangle \\
=&\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2 - \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle - \|v\|^2 \\
=&4\langle u, v\rangle
\end{aligned}
$$

25 Since S is not injective, there exists a nonzero vector $u \in V$ such that $Su = 0$. Then $\langle u, u\rangle_1 = \langle Su, Su\rangle = 0$, which means $\langle , \rangle_1$ is not an inner product.

Exercise 6.B

2* By Property 6.35, we can extend the orthogonal list $e_1, \cdots, e_m$ to an orthogonal basis $e_1, \cdots, e_m, e_{m+1}, \cdots, e_n$ of V, where $m \leq n =$dim$V$. Thus $v = a_1 e_1 + \cdots + a_n e_n$ with $a_i = \langle v, e_i\rangle$ and $\|v\|^2 = |a_1|^2 + \cdots + |a_n|^2$.
Thus we have that $\|v\|^2 = |a_1|^2 + \cdots + |a_m|^2$ if and only if $a_{m+1} = \cdots = a_n = 0$, which is true if and only if $v \in \text{span}\{e_1, \cdots e_m\}$.

4 It is obvious that $\|\frac{1}{\sqrt{2\pi}}\| = 1$ and for any $j, k = 1, \cdots, n$,

$$\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos kx}{\sqrt{\pi}}\, dx = 0$$
$$\langle \frac{1}{\sqrt{2\pi}}, \frac{\sin kx}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin kx}{\sqrt{\pi}}\, dx = 0$$
$$\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{\sin jx}{\sqrt{\pi}} \frac{\cos kx}{\sqrt{\pi}}\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin((j+k)x) + \sin((j-k)x)}{2}\, dx = 0$$
$$\langle \frac{\sin jx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{\sin jx}{\sqrt{\pi}} \frac{\sin kx}{\sqrt{\pi}}\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos((j-k)x) - \cos((j+k)x)}{2}\, dx = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$$
$$\langle \frac{\cos jx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \rangle = \int_{-\pi}^{\pi} \frac{\cos jx}{\sqrt{\pi}} \frac{\cos kx}{\sqrt{\pi}}\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos((j-k)x) + \cos((j+k)x)}{2}\, dx = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$$

5 Denote $v_1 = 1, v_2 = x, v_3 = x^2$, then by the Gram-Schmidt Procedure,

$$e_1 = \frac{v_1}{\|v_1\|} = 1,$$
$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = 2\sqrt{3}(x - \frac{1}{2}),$$
$$e_3 = \frac{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}{\|v_2 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2\|} = \frac{x^2 - x + \frac{1}{6}}{\|x^2 - x + \frac{1}{6}\|} = 6\sqrt{5}(x^2 - x + \frac{1}{6})$$

7 If we define $\varphi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ by $\varphi(p) = p(\frac{1}{2})$, then Riesz representation theorem guarantees that there exists a unique $q \in \mathcal{P}_2(\mathbb{R})$ such that $\varphi(p) = \langle p, q \rangle$ for all $p \in \mathcal{P}_2(\mathbb{R})$. The proof of Riesz representation theorem gives an explicit way to construct q. First, we pick an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, say $\{e_1, e_2, e_3\}$ where

$$e_1 = 1, \quad e_2 = \sqrt{3}(2x - 1), \quad e_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right).$$

Then, for a real vector space, we have

$$q = \varphi(1)1 + \varphi(\sqrt{3}(2x-1))\sqrt{3}(2x-1) + \varphi(\sqrt{180}(x^2-x+\frac{1}{6}))\sqrt{180}(x^2-x+\frac{1}{6}) = -15x^2 + 15x - \frac{3}{2}.$$

9* If we apply the Gram-Schmidt Procedure on linearly depend list, say $\{v_1, v_2\}$ with $v_2 = \lambda v_1$, then we have $v_2 - \langle v_2, \frac{v_1}{\|v_1\|} \rangle \frac{v_1}{\|v_1\|} = \lambda v_1 - \lambda v_1 = 0$.

14 It suffices to prove that $\{v_1, \cdots, v_n\}$ is linearly independent. Assume that $a_1 v_1 + \cdots + a_n v_n = 0$, then $a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n) = a_1 e_1 + \cdots + a_n e_n$. Taking the norm of each side leads to

$$\|a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n)\|^2 = \|a_1 e_1 + \cdots + a_n e_n\|^2 = a_1^2 + \cdots + a_n^2$$

But by the Triangle Inequality 6.18, the right hand side

$$\|a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n)\| \leq |a_1|\|e_1 - v_1\| + \cdots + |a_n|\|e_n - v_n\|$$

Thus if $a_1 = a_2 = \cdots = a_n = 0$ does not hold, then there holds

$$a_1^2 + \cdots + a_n^2 \leq (|a_1|\|e_1 - v_1\| + \cdots + |a_n|\|e_n - v_n\|)^2 < \frac{1}{n}\left(|a_1| + \cdots + |a_n|\right)^2$$

which contradicts with Q12 in Ex 6.A.

Exercise 6.C

4 First, we extend $\{(1,2,3,-4),(-5,4,3,2)\}$ to some basis of U, say $\beta = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -5 \\ 4 \\ 3 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We then apply the Gram-Schmidt process to β to get an orthonormal basis $\{u_1, u_2, u_3, u_4\}$.

$$\begin{aligned}
u_1 &= \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{30}}(1,2,3,-4) \\
u_2 &= \frac{v_2 - \langle v_2, u_1\rangle u_1}{\|v_2 - \langle v_2, u_1\rangle u_1\|} = \frac{1}{\sqrt{12030}}(-77,56,39,38) \\
u_3 &= \frac{v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2}{\|v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2\|} = \frac{1}{\sqrt{92230}}(60,-153,230,111) \\
u_4 &= \frac{v_4 - \langle v_4, u_1\rangle u_1 - \langle v_4, u_2\rangle u_2 - \langle v_4, u_3\rangle u_3}{\|v_4 - \langle v_4, u_1\rangle u_1 - \langle v_4, u_2\rangle u_2 - \langle v_4, u_3\rangle u_3\|} = \frac{1}{\sqrt{230}}(10,9,0,7)
\end{aligned}$$

5 By Property 6.47, any $v \in V$ can be written as $v = v_1 + v_2$ where $v_1 \in U,\ v_2 \in U^{\perp}$. Then by Properties 6.55 (b), we have that $v_1 = P_U v,\ v_2 = P_{U^\perp} v$, which means $v = P_U v + P_{U^\perp} v$.

6*($\Rightarrow$) : If $P_U P_W = 0$, then for all $u \in U,\ w \in W,\quad \langle u, w\rangle = \langle P_W u + P_{W^\perp} u, P_U w + P_{U^\perp} w\rangle$. As $P_W u + P_{W^\perp} u \in U$ and $P_U w + P_{U^\perp} w \in W$, we have that

$$\langle u, w\rangle = \langle P_W u, P_U w\rangle = \langle (P_U + P_{U^\perp}) P_W u, P_U w\rangle = \langle P_U P_W u, P_U w\rangle = 0.$$

($\Leftarrow$): If for all $u \in U,\ w \in W, \langle u, w\rangle = 0$, then for any $v \in V,\ P_U P_W v \in U \cap W$, then $\|P_U P_W v\|^2 = \langle P_U P_W v, P_U P_W v\rangle = 0$, then $P_U P_W v = 0$.

7 We show that $U = \operatorname{range} P$ and P is the orthogonal projection of V onto the range of P. Firstly we show that $\ker P \oplus \operatorname{range} P$. Note that

$$v = Pv + (I - P)v,$$

with $(I-P)v \in \ker P$ (because $P(I-P)v = (P - P^2)v = 0$) and $Pv \in \operatorname{range} P$. This shows that $V = \ker P + \operatorname{range} P$. In addition, $\ker P \cap \operatorname{range} P = \{0\}$ because for $u \in \ker P \cap \operatorname{range} P$, we have $u = Pu'$ for some $u' \in V$ as $u \in \operatorname{range} P$, then

$$u = Pu' = P^2 u' = P(Pu') = Pu = 0$$

as $u \in \ker P$. This shows that $V = \ker P \oplus \operatorname{range} P$.
Secondly combined with that every vector in $\ker P$ is orthogonal to every vector in $\operatorname{range} P$, then we have that $\ker P = (\operatorname{range} P)^{\perp}$. Then $V = \operatorname{range} P \oplus (\operatorname{range} P)^{\perp}$ and $P = P_U$ is the orthogonal projection of V onto $\operatorname{range} P$.

8* We show that $U = \operatorname{range} P$ and P is the orthogonal projection of V onto the range of P. Note that same to the proof of Q7, we have $V = \ker P \oplus \operatorname{range} P$.
Then we show that $\ker P = (\operatorname{range} P)^{\perp}$. Note that we have $V = \operatorname{range} P \oplus (\operatorname{range} P)^{\perp}$. If

we can show that $\ker P \subset (\operatorname{range} P)^{\perp}$, then, by the dimension, we have $\ker P = (\operatorname{range} P)^{\perp}$. The claim is true because for any $w \in \operatorname{range} P$, we have $w = Pw'$ for some $w' \in V$, but $w = Pw' = P^2w' = P(Pw') = Pw$. So, for any $u \in \ker P$, we have

$$\|w\| = \|Pw\| = \|P(w + au)\| \leq \|w + au\|$$

for any $a \in \mathbb{F}$. After expanding the inner products, we have $0 \leq \bar{a} \langle w, u\rangle + a \langle u, w\rangle + |a|^2 \langle u, u\rangle$. In particular, we take $a = -\frac{\langle w,u\rangle}{\langle u,u\rangle}$ to obtain $\langle u, w\rangle = 0$, which means $u \in (\operatorname{range} P)^{\perp}$ since w is arbitrary.

Finally, we have $P = P_U$ as an orthogonal projection of V onto $\operatorname{range} P$.

11 Denote $v = (1, 2, 3, 4)$, then

$$\|P_U v - v\| = \min_{u\in U} \|u - v\|$$

It is easy to find an orthogonal basis $\{e_1, e_2\} = \{\frac{1}{\sqrt{2}}(1, 1, 0, 0), \frac{1}{\sqrt{5}}(0, 0, 1, 2)\}$ of U. Thus we have

$$P_U v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 = \frac{3}{\sqrt{2}}e_1 + \frac{11}{\sqrt{5}}e_2 = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$$

12 Denote $U = \{p(x) \in \mathcal{P}_3(\mathbb{R}) : p(0) = p'(0) = 0\} = \operatorname{span}\{x^2, x^3\}$, and $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ to be the usual inner product. Similarly, we have that $P_U(2 + 3x)$ is the polynomial to achieve the minimum value.

Using the Gram-Schmidt Procedure on the basis $\{x^2, x^3\}$, we can find an orthogonal basis $\{e_1, e_2\} = \{\sqrt{5}x^2, 6\sqrt{7}(x^3 - \frac{5}{6}x^2)\}$ of U, hence we have

$$P_U(2 + 3x) = \langle 2 + 3x, e_1\rangle e_1 + \langle 2 + 3x, e_2\rangle e_2 = \frac{85}{12}x^2 - \frac{203}{60}(6x^3 - 5x^2) = -\frac{203}{10}x^3 + 24x^2.$$

14* (a) For any $g \in U^{\perp}$, we have that $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\ dx = 0,\ \forall\ f \in U$.

If $g(x)$ is not zero, by the continuity, there exists a positive number $\epsilon > 0$ and a nonzero point $x_0 \in (-1, 1)$ such that $|g(x_0)| > \epsilon$. And then there exists a small enough and positive number δ such that for any $x \in (x_0-\delta, x_0+\delta)$ which does not contain 0, $|g(x)| > \epsilon/2$. Thus we can choose a sequence of functions $f_n(x) \in U$, such that $f_n(x)$ converges to the function

$$f(x) = \begin{cases} 1, & \text{if } x \in [x_0 - \delta, x_0 + \delta] \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that

$$\int_{-1}^{1} f(x)g(x)\ dx = \lim_{n\to+\infty} \int_{-1}^{1} f_n(x)g(x)\ dx = 0$$

But

$$\int_{-1}^{1} f(x)g(x)\ dx = \int_{x_0-\delta}^{x_0+\delta} g(x)\ dx \geq \frac{\epsilon}{2} \times 2\delta > 0$$

which is a contradiction.

(b) By (a), we have that $C_R([-1, 1]) \neq U \oplus U^{\perp}$. And $(U^{\perp})^{\perp} = 0^{\perp} = C_R([-1, 1]) \neq U$.