MATH 2040C Linear Algebra II

2017-18 Term 2

Midterm 1

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NAME: Solution ID: __________

Instruction: Answer ALL questions and show your work with explanation.

Time: 60 minutes

Question	Score
1	
2	
3	
4	
5	
Total	/40

1. (**True or False**) Please circle the correct answer. Each question is worth 1 point.

(a) Suppose U and W are both four-dimensional subspaces of $\mathbb{R}^7$. Then $U \cap W$ is not the zero subspace.

TRUE FALSE

(b) If S is a linearly independent subset of a vector space V, and $v \in V$ is a vector such that $S \cup \{v\}$ is linearly dependent, then $v \in \operatorname{span} S$.

TRUE FALSE

(c) The real vector space $\mathbb{R}^{(0,1)} = \{f : (0,1) \to \mathbb{R}\}$ of all real-valued functions defined on $(0,1)$, with the standard addition and scalar multiplication, do not have a basis.

Every vector space has a basis

TRUE FALSE

(d) Let $T : V \to W$ be a linear map between vector spaces V and W. Then the range of T is a subspace of W.

TRUE FALSE

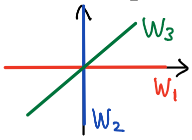
(e) The real vector space of all 4×4 skew-symmetric matrices has dimension 10.

$\dim = 6$

TRUE FALSE

(f) Let V be a complex vector space and W_1, W_2, W_3 be its subspaces with intersection $W_1 \cap W_2 \cap W_3 = \{0\}$. Then $W_1 + W_2 + W_3$ is a direct sum.

Counter example :



TRUE FALSE

(g) Let V be a n-dimensional vector space and $S \subset V$ be a linearly independent subset consisting of n vectors. Then S is a basis of V.

TRUE FALSE

(h) The vector space $\mathbb{R}^\infty = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\}$ of all sequences of real numbers has a basis $\{e_1, e_2, \ldots\}$, where e_k is the sequence with the k-th term equals to 1 and all other terms equal to 0.

TRUE FALSE

$\operatorname{span}\{e_1, e_2, \cdots\}$

$= \{(x_1, x_2, \cdots) \in \mathbb{R}^\infty : \exists n \text{ s.t. } x_i = 0 \ \forall i \geq n\}$

= subspace of all sequences with finitely many non-zero terms

2. (12 pts) Answer the following questions.

(a) Consider $\mathbb{C}^2$ as a vector space over $\mathbb{F} = \mathbb{C}$. Express $(2, 0) \in \mathbb{C}^2$ as a linear combination of the vectors $(1, i)$ and $(i, 1)$ or show that it is impossible.

$$(2,0) = (1, i) + (-i)(i, 1)$$

Rmk One may also find the coefficients using Gaussian elimination:

Suppose $a(1,i) + b(i,1) = (2,0)$

then $\begin{cases} a + ib = 2 \\ ia + b = 0 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & i & 2 \\ i & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 2 \\ 0 & 2 & -2i \end{array}\right] \quad R_2 \to R_2 - iR_1$$

$$\sim \left[\begin{array}{cc|c} 1 & i & 2 \\ 0 & 1 & -i \end{array}\right]$$

$\therefore$ $b = -i$

$a + ib = 2 \Rightarrow a = 2 - i(-i) = 1$

(b) Is the subset $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ a subspace of the vector space $\mathbb{R}^2$? Justify your answer.

No. Let $S = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$

Note that $(1)(0) = (0)(1) = 0$

$\Rightarrow (1,0), (0,1) \in S$

but $(1)(1) = 1 \neq 0$

$\Rightarrow (1,0) + (0,1) = (1,1) \notin S$

$\therefore$ S is not closed under addition

$\therefore$ S is not a subspace.

(c) Is $\{1+x, 1+2x+3x^2, x+2x^2\}$ a linearly independent subset of the vector space of all real polynomials?

Suppose

zero polynomial

$$c_1(1+x)+c_2(1+2x+3x^2)+c_3(x+2x^2)=0$$

then $(c_1+c_2)+(c_1+2c_2+c_3)x+(3c_2+2c_3)x^2=0$

Comparing coefficients

$$\Rightarrow \begin{cases} c_1+c_2 = 0 \\ c_1+2c_2+c_3=0 \\ 3c_2+2c_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$\Rightarrow c_1=c_2=c_3=0$

$\Rightarrow$ the subset is linearly independent.

3. (7 pts) Let $\mathcal{P}_2(\mathbb{R})$ be the real vector space of all real polynomials of degree at most 2 and $U = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p'(1) = 0\}$ be its subspace.

(a) Find a basis S of U;

(b) Extend the set S in part (a) to a basis of $\mathcal{P}_2(\mathbb{R})$;

(c) Find a subspace W of $\mathcal{P}_2(\mathbb{R})$ such that $\mathcal{P}_2(\mathbb{R}) = U \oplus W$.

a. Note $x \notin U \Rightarrow U \neq P_2(\mathbb{R})$

$\therefore \dim U < \dim P_2(\mathbb{R}) = 3$

Let $S = \{1, (x-1)^2\} \subset U$. S is lin. indept

$\therefore \dim U \geq 2$

$\therefore \dim U = 2$ and S is a basis.

<u>Alt soln</u> Let $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$

$p'(x) = b + 2cx$

$p'(1) = b + 2c$

$\therefore p(x) \in U \iff b + 2c = 0$

$$\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -2c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

We can take $S = \{1, -2x + x^2\}$

b. $x \notin U = \operatorname{span} S$

$\therefore S' = \{1, (x-1)^2, x\}$ is lin indept, $|S'| = 3$

$\dim P_2(\mathbb{R}) = 3 \Rightarrow S'$ is a basis of $P_2(\mathbb{R})$

c. We can take $W = \operatorname{span}\{x\}$.

$= \{kx : k \in \mathbb{R}\}$

4. (8 pts) Let V be a vector space and $T : V \to V$ be a linear map.

   (a) Show that $W = \{v \in V : T(v) = v\}$ is a subspace of V.

   (b) Suppose that $T^2 = T$. Show that $V = \operatorname{null} T \oplus W$.

a. ① $T(\vec{0}) = \vec{0}$ because T is linear. Hence $\vec{0} \in W$

② If $w_1, w_2 \in W$, then $T(w_1 + w_2) = T(w_1) + T(w_2)$

$= w_1 + w_2$

$\Rightarrow w_1 + w_2 \in W$

③ If $w \in W$, $\lambda \in \mathbb{F}$, then $T(\lambda w) = \lambda T(w)$

$= \lambda w$

$\Rightarrow \lambda w \in W$

$\therefore W$ is a subspace.

6. $\forall\, v \in V,\quad v = (v - T(v)) + T(v)$

Note that $T(v - T(v)) = T(v) - T^2(v) = \vec{0}$

and $T(T(v)) = T^2(v) = T(v)$

$\therefore\ v - T(v) \in \operatorname{null} T,\quad T(v) \in W$

$\therefore\ V = \operatorname{null} T + W$

To show it is a direct sum, suppose $v \in \operatorname{null} T \cap W$

then $v = T(v) = 0 \Rightarrow \operatorname{null} T \cap W = \{0\}$

$\therefore\ V = \operatorname{null} T \oplus W$

5. (5 pts) Let V be a real vector space, W_1 and W_2 be its subspaces of dimension 2. Suppose $\{v_0\}$ is a basis of $W_1 \cap W_2$, $\{v_0, v_1\}$ is a basis of W_1 and $\{v_0, v_2\}$ is a basis of W_2. Prove that $\{v_0, v_1, v_2\}$ is linearly independent from the definition.

Suppose $c_0, c_1, c_2 \in \mathbb{R}$ and

$$c_0 v_0 + c_1 v_1 + c_2 v_2 = \vec{0}.$$

Then $c_0 v_0 + c_1 v_1 = -c_2 v_2 \in W_2$

But $c_0 v_0 + c_1 v_1 \in W_1$

$\Rightarrow$ $-c_2 v_2 \in W_1 \cap W_2 = \text{span}\{v_0\}$

Hence $-c_2 v_2 = k v_0$ for some $k \in \mathbb{R}$

$\Rightarrow$ $k v_0 + c_2 v_2 = \vec{0}$

$\{v_0, v_2\}$ is lin indept $\Rightarrow$ $c_2 = 0$

$\Rightarrow$ $c_0 v_0 + c_1 v_1 = -c_2 v_2 = \vec{0}$

$\{v_0, v_1\}$ is lin indept $\Rightarrow$ $c_0 = c_1 = 0$

$\therefore$ $c_0 = c_1 = c_2 = 0 \Rightarrow \{v_0, v_1, v_2\}$ is lin. indept

—END OF TEST 1—