MATH2010 Advanced calculus, 2020-21
MIDTERM
Suggested Solution

1. (a) $\vec{AB} = (1, 1, -2)$
$|AB| = \sqrt{6}$
$\vec{AC} = (-1, 0, 2)$
$|AC| = \sqrt{5}$
$AB \cdot AC = (1, 1, -2) \cdot (-1, 0, 2) = -5$
$\cos\theta = \dfrac{AB \cdot AC}{|AB| \cdot |AC|} = -\dfrac{5}{\sqrt{30}}$
$\theta = \cos^{-1}\left(-\dfrac{\sqrt{30}}{6}\right) \approx 155.9^\circ$ (or 2.72 radian or 0.866π radian)

(b) $\vec{AD} = (3, -2, 0)$
Volume of the parallelepiped spanned by AB, AC, AD
$= \det\begin{pmatrix} 1 & 1 & -2 \\ -1 & 0 & 2 \\ 3 & -2 & 0 \end{pmatrix}$
$= 0 + 6 - 4 + 4 - 0 - 0$
$= 6$
Volume of the tetrahedron $= \dfrac{1}{6} \cdot 6 = 1$

(c) $n = \vec{AB} \times \vec{AC}$
$= (1, 1, -2) \times (-1, 0, 2)$
$= (2, 0, 1)$ is a normal vector to the plane required.
$n \cdot A = (2, 0, 1) \cdot (1, 2, 1) = 3$
Hence, $2x + 0y + z = 3$ is an equation of the plane.

2. (a) $f(x, y) = x^2y^2 + 2x$
$f_x(x, y) = 2xy^2 + 2$
$f_x(2, 1) = 6$
$f_y(x, y) = 2x^2y$
$f_y(2, 1) = 8$
Equation of tangent plane at $2, 1, 8$ is:
$z = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1)$
$= 8 + 6(x - 2) + 8(y - 1)$
$= 6x + 8y - 12$

(b) $r\left(\dfrac{\pi}{4}\right) = \left(1, 1, \dfrac{\pi}{4}\right)$
$r'(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t, 1)$
$r'\left(\dfrac{\pi}{4}\right) = (-1, 1, 1)$
Hence a parametric equation for the tangent line is $r(t) = \left(1, 1, \dfrac{\pi}{4}\right) + t(-1, 1, 1)$

(c) A normal vector of the tangent plane in part (a) is $(6,8,-1)$.
$(6,8,-1)$ is not parallel to $(-1,1,1)$, so the line and the plane are NOT perpendicular.
$(6,8,-1)\cdot(-1,1,1)\neq 0$, so the line and the plane are NOT parallel.

3. $C_1 : r = 4\cos\theta$

$r^2 = 4r\cos\theta$

$x^2+y^2 = 4x$ ...(1)

$C_2 : r = \dfrac{2}{\sin\theta+\cos\theta}$

$r(\sin\theta+\cos\theta) = 2$

$x+y=2$ ...(2)

Put (2) into (1), $x^2+(2-x)^2 = 4x$

$2x^2-8x+4=0$

$2(x-2)^2=4$

$x = 2\pm\sqrt{2}$

$y = \mp\sqrt{2}$

The intersection points are $(2+\sqrt{2},-\sqrt{2}),(2-\sqrt{2},\sqrt{2})$

Other possible forms of answers:

$\left(4\cos^2\left(\frac{3\pi}{8}\right), 2\sin\left(\frac{3\pi}{4}\right)\right), \left(4\cos^2\left(\frac{7\pi}{8}\right), 2\sin\left(\frac{7\pi}{4}\right)\right)$

4. (a) $\displaystyle\lim_{(x,y)\to(1,1)}\frac{(x^2-y^2)(x-y)}{\sqrt{(x-1)^2+(y-1)^2}} = \lim_{(x,y)\to(1,1)}\frac{(x+y)(x-y)^2}{\sqrt{(x-1)^2+(y-1)^2}}$
$\displaystyle = \lim_{(x,y)\to(1,1)}\frac{2(x-y)^2}{\sqrt{(x-1)^2+(y-1)^2}} = \lim_{r\to0}\frac{2((1+r\cos\theta)-(1+r\sin\theta))^2}{r}$
$\displaystyle = \lim_{r\to0}2r(\cos\theta-\sin\theta)^2 = 0$, by the Squeeze Theorem.

(b) Note that

$$\lim_{\substack{(x,y,z)\to(x_0,0,0)\\x=kz^2,y=z}}\frac{xyz}{x^2+y^4+z^4} = \lim_{\substack{(x,y,z)\to(x_0,0,0)\\x=kz^2,y=z}}\frac{kz^4}{(k^2+2)z^4} = \frac{k}{k^2++2}$$

Therefore, the value depends on k and hence the limit does not exist.

(c) $-1 \le |\cos\dfrac{x}{x^2+y^2}| \le 1$

$\dfrac{1}{2} \le \dfrac{1}{\sqrt{3+\cos\frac{x}{x^2+y^2}}} \le \dfrac{1}{\sqrt{2}}$

$$0 = \frac{1}{2}\lim_{(x,y)\to(0,0)}\sin(x-y) \le \lim_{(x,y)\to(0,0)}\frac{\sin x - y}{\sqrt{3+\cos\frac{x}{x^2+y^2}}} \le \frac{1}{\sqrt{2}}\lim_{(x,y)\to(0,0)}\sin(x-y) = 0$$

By the Squeeze Theorem, this limit is 0.

Remark: Two paths test is usually used to check the function is NOT differentiable at some point. Note that we cannot easily conclude the function is differentiable even if the limits exist along many paths.

5. Case 1: $xyz \neq 0$.
Note that f_x, f_y and f_z exist and are continuous in this region. Hence f is differentiable.

Case 2: Exactly one of x, y or z is zero.
Say $x = 0$ and check the differentiability of f at $(0, y_0, z_0)$ for $y_0 z_0 \neq 0$. Indeed,

$$\lim_{h \to 0} \frac{f(h, y_0, z_0) - f(0, y_0, z_0)}{h} = \lim_{h \to 0} \frac{\sqrt{|h y_0 z_0|}}{h} = \sqrt{y_0 z_0} \lim_{h \to 0} \frac{1}{\sqrt{|h|}}$$

Hence f_x doen NOT exist. Hence f is not diffenrentiable in this case.

Case 3: Exactly two of of x, y or z are zero.
Say $x_0 \neq 0$ and check the differentiability of f at $(x_0, 0, 0)$. Note that

$$f_x(x_0, 0, 0) = \lim_{h \to 0} \frac{f(x_0 + h, 0, 0) - f(x_0, 0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Similarly, we have $f_y(x_0, 0, 0) = f_z(x_0, 0, 0) = 0$. Then we compute the error

$$\begin{aligned} \epsilon(x, y, z) &= f(x, y, z) - f(x_0, 0, 0) - f_x(x_0, 0, 0)(x - x_0) - f_y(x_0, 0, 0)y - f_z(x_0, 0, 0)z \\ &= \sqrt{|xyz|} - 0 = \sqrt{|xyz|} \end{aligned}$$

Now we claim that the the following limit does not exist.

$$\lim_{(x,y,z) \to (x_0,0,0)} \frac{\epsilon(x, y, z)}{\sqrt{(x - x_0)^2 + y^2 + z^2}} = \lim_{(x,y,z) \to (x_0,0,0)} \frac{\sqrt{|xyz|}}{\sqrt{(x - x_0)^2 + y^2 + z^2}}$$

Consider two paths test:

$$\begin{aligned} \lim_{\substack{(x,y,z) \to (x_0,0,0) \\ x = x_0, y = 0}} \frac{\sqrt{|xyz|}}{\sqrt{(x - x_0)^2 + y^2 + z^2}} &= \lim_{\substack{(x,y,z) \to (x_0,0,0) \\ x = x_0, y = 0}} \frac{0}{\sqrt{0 + z^2}} = 0 \\ \lim_{\substack{(x,y,z) \to (x_0,0,0) \\ x = x_0, y = z}} \frac{\sqrt{|xyz|}}{\sqrt{(x - x_0)^2 + y^2 + z^2}} &= \lim_{\substack{(x,y,z) \to (x_0,0,0) \\ x = x_0, y = z}} \frac{\sqrt{|x_0 y^2|}}{\sqrt{0 + 2y^2}} = \sqrt{\frac{|x_0|}{2}} \neq 0 \end{aligned}$$

Hence the f cannot be linearly approximated and thus not differentiable at $(x_0, 0, 0)$.

Case 4: $x = y = z = 0$.
Note that

$$f_x(0,0,0) = \lim_{h\to 0} \frac{f(h,0,0) - f(x_0,0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, we have $f_y(0,0,0) = f_z(0,0,0) = 0$. Then we compute the error

$$\begin{aligned}
\epsilon(x,y,z) &= f(x,y,z) - f(0,0,0) - f_x(0,0,0)x - f_y(0,0,0)y - f_z(0,0,0)z \\
&= \sqrt{|xyz|} - 0 = \sqrt{|xyz|}
\end{aligned}$$

Using spherical coordinates, i.e. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$,

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \frac{\epsilon(x,y,z)}{\sqrt{x^2+y^2+z^2}} &= \lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{|xyz|}}{\sqrt{x^2+y^2+z^2}} \\
&= \lim_{r\to 0} \sqrt{\frac{|r^3 \sin^2\theta \sin\phi\cos\phi\cos\theta|}{r^2}} \\
&= \lim_{r\to 0} \sqrt{r}\sqrt{|\sin^2\theta\sin\phi\cos\phi\cos\theta|} \\
&\leq \lim_{r\to 0} \sqrt{r} = 0.
\end{aligned}$$

We use Squeeze Theorem in the last inequality. Hence f is differentiable at (0,0,0).

To conclude, f is differentiable exactly on $\{(x,y,z) | xyz \neq 0 \text{ or } x = y = z = 0\}$.