MATH 2010B Advanced Calculus I
(2014-2015, First Term)
Quiz 2
Suggested Solution

1. (a) Method 1:

$$\begin{aligned} xz + y^2 &= 0 \\ (x \quad y \quad z)\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= 0 \end{aligned}$$

Let

$$M = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Then

$$\begin{aligned} \det(M - \lambda I) &= 0 \\ \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)(1-\lambda) &= 0 \\ \lambda &= 1, \frac{1}{2}, -\frac{1}{2} \end{aligned}$$

And the corresponding orthonormal eigenvectors are

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

Let

$$R = [v_1 \ v_2 \ v_3]^t = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

And take the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ (u+w)/\sqrt{2} \\ (u-w)/\sqrt{2} \end{pmatrix}$$

Then we have

$$u^2 + \frac{1}{2}v^2 - \frac{1}{2}w^2 = 0$$

which is a cone.

Method 2:

$$\begin{aligned} xz + y^2 &= 0 \\ 4xz + 4y^2 &= 0 \\ (x+z)^2 - (x-z)^2 + 4y^2 &= 0 \end{aligned}$$

Take $u = x + z,\ v = x - z,\ w = y$, then it becomes

$$\begin{aligned} u^2 - v^2 + 4w^2 &= 0 \\ u^2 + 4w^2 &= v^2 \end{aligned}$$

Which is a cone.

(b) Put $z = x + y + 1$ into $xz + y^2 = 0$, then

$$\begin{aligned} x(x+y+1) + y^2 &= 0 \\ x^2 + xy + x + y^2 &= 0 \\ (x \quad y)\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 0 \end{aligned}$$

Take

$$M = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \det(M - \lambda I) &= 0 \\ \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right) &= 0 \\ \lambda &= \frac{1}{2}, \frac{3}{2} \end{aligned}$$

Thus $\lambda_1 \lambda_2 > 0 \Rightarrow$ ellipse.

2. <u>Method 1</u>: Take the polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{|r\cos\theta|}{\sqrt{r^2}} \\ &= \lim_{r\to 0} \frac{|r||\cos\theta|}{|r|} \\ &= \lim_{r\to 0} |\cos\theta| \\ &= |\cos\theta| \end{aligned}$$

Therefore the limit does not exist since $|\cos\theta|$ varies for different value of θ.

<u>Method 2</u>: Consider the limit along $x = 0$ and $y = 0$.

Along $x = 0$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} \frac{|y|}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{|y|}{\sqrt{y^2}} = 1.$$

Along $y = 0$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} \frac{|y|}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{0}{\sqrt{x^2}} = 0.$$

Since the limits along two different paths are not the same, the limit does not exist.

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