MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 9
Suggested Solution

3. $f(x,y) = e^{-x^2-y^2};\ P(0,0)$.
$\nabla f =< f_x, f_y >=< -2xe^{-x^2-y^2}, -2ye^{-x^2-y^2} >;\ \nabla f(0,0) =< 0, 0 >$.

9. $f(x,y,z) = 2\sqrt{xyz};\ P(3,-4,-3)$.
$\nabla f =< f_x, f_y, f_z >=< \frac{yz}{\sqrt{xyz}}, \frac{xz}{\sqrt{xyz}}, \frac{xy}{\sqrt{xyz}} >;$
$\nabla f(3,-4,3) =< \frac{12}{\sqrt{36}}, \frac{-9}{\sqrt{36}}, \frac{-12}{\sqrt{36}} >=< 2, -\frac{3}{2}, -2 >$

15. $f(x,y) = \sin x \cos y;\ P(\pi/3, -2\pi/3),\ \vec{v} =< 4, -3 >$.
$\nabla f =< f_x, f_y >=< \cos x \cos y, -\sin x \sin y >;\ \vec{u} = \frac{\vec{v}}{|\vec{v}|} =< \frac{4}{5}, -\frac{3}{5} >;$
$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \frac{4}{5}\cos x \cos y - \frac{3}{5}\sin x \sin y;$
$D_{\vec{u}}f(P) = \frac{4}{5}\cos(\pi/3)\cos(-2\pi/3) - \frac{3}{5}\sin(\pi/3)\sin(-2\pi/3) = -\frac{4}{20} - \frac{9}{20} = -\frac{13}{20}.$

19. $f(x,y,z) = e^{xyz};\ P(4,0,-3),\ \vec{v} = \vec{j} - \vec{k} =< 0, 1, -1 >$.
$\nabla f =< f_x, f_y, f_z >=< yze^{xyz}, xze^{xyz}, xye^{xyz} >;\ \vec{u} = \frac{\vec{v}}{|\vec{v}|} =< 0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} >;$
$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \frac{\sqrt{2}}{2}xze^{xyz} - \frac{\sqrt{2}}{2}xye^{xyz};$
$D_{\vec{u}}f(P) = -6\sqrt{2}.$

23. $f(x,y) = \ln(x^2+y^2);\ P(3,4)$.
$\nabla f =< \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} >;\ \nabla f(3,4) =< \frac{6}{25}, \frac{8}{25} >;$
Then maximum directional derivative of f at P is $|\nabla f(3,4)| = \frac{2}{5}$ and the direction in which it occurs is $< \frac{6}{25}, \frac{8}{25} >$.

27. $f(x,y,z) = \sqrt{xy^2z^3};\ P(2,2,2)$.
$\nabla f =< \frac{y^2z^3}{2\sqrt{xy^2z^3}}, \frac{2xyz^3}{2\sqrt{xy^2z^3}}, \frac{3xy^2z^2}{2\sqrt{xy^2z^3}} >;\ \nabla f(2,2,2) =< 2, 4, 6 >;$
Then maximum directional derivative of f at P is $|\nabla f(2,2,2)| = 2\sqrt{14}$ and the direction in which it occurs is $< 2, 4, 6 >$.

33. $x^{1/3} + y^{1/3} + z^{1/3} = 1;\ P(1,-1,1)$.
Let $F(x,y,z) = x^{1/3} + y^{1/3} + z^{1/3} - 1;$
$F(1,-1,1) = 0 \Rightarrow P(1,-1,1)$ lies on the plane.
$\nabla F =< \frac{1}{3}x^{-2/3}, \frac{1}{3}y^{-2/3}, \frac{1}{3}z^{-2/3} >;\ \nabla F(1,-1,1) =< \frac{1}{3}, \frac{1}{3}, \frac{1}{3} >$. The equation of the tangent plane is

$$\nabla F(1,-1,1) \cdot < x-1, y+1, z-1 >= 0 \Rightarrow x + y + z = 1$$

43. $z = Ax^2 + By^2$;
Let $F(x, y, z) = z - Ax^2 - By^2$;
Since (x_0, y_0, z_0) is on the paraboloid, therefore $z_0 = Ax_0^2 + By_0^2$;
$\nabla F =< -2Ax, -2By, 1 >$; $\nabla F(x_0, y_0, z_0) =< -2Ax_0, -2By_0, 1 >$.
The equation of the tangent plane is

$$\begin{aligned}
\nabla F(x_0, y_0, z_0)\cdot < x - x_0, y - y_0, z - z_0 > &= 0 \\
-2Ax_0x - 2By_0y + z + 2Ax_0^2 + 2By_0^2 - z_0 &= 0 \\
-2Ax_0x - 2By_0y + z + 2z_0 - z_0 &= 0 \\
-2Ax_0x - 2By_0y + z + z_0 &= 0 \\
z + z_0 &= 2Ax_0x + 2By_0y
\end{aligned}$$

49. $z = f(x, y); f(x, y) = \dfrac{1}{10}\left(x^2 - xy + 2y^2\right)$

(a) Let $F(x, y, z) = \dfrac{1}{10}\left(x^2 - xy + 2y^2\right) - z$;
$F(2, 1, 0.4) = \dfrac{4}{10} - 0.4 = 0 \Rightarrow P(2, 1, 0.4)$ lies on the plane.
$\nabla F =< \dfrac{1}{10}(2x - y), \dfrac{1}{10}(-x + 4y), -1 >$; $\nabla F(2, 1, 0.4) =< 0.3, 0.2, -1 >$. The equation of the tangent plane is

$$\nabla F(2, 1, 0.4)\cdot < x-2, y-1, z-0.4 >= 0 \Rightarrow 0.3x+0.2y-z-0.4 = 0 \Rightarrow z = 0.3x+0.2y-0.4$$

(b) $\nabla f =< \dfrac{1}{10}(2x - y), \dfrac{1}{10}(-x + 4y) >$; $\nabla f(2, 1) = (0.3, 0.2)$;
Linear approximation $L(2.2, 0.5) = f(2, 1) + \nabla f(2, 1) < 2.2 - 2, 0.9 - 1 >= 0.4 + (0.3)(0.2) + (0.2)(-0.1) = 0.44$. Actual altitude at this point is $f(2.2, 0.9) = 0.448$.

51.

$$\begin{cases} z^2 &= x^2 + y^2 \\ 2x + 3y + 4z + 2 &= 0 \end{cases}$$

Note that $(-5)^2 = (3)^2 + (4)^2$ and $2(3) + 3(4) + 4(-5) + 2 = 0$. Therefore, $P(3, 4, -5)$ lies on the intersection of the ellipse.
Let $F(x, y, z) = x^2 + y^2 - z^2$ and $G(x, y, z) = 2x + 3y + 4z + 2$.
$\nabla F =< 2x, 2y, -2z >$ and $\nabla G =< 2, 3, 4 >$;
$\nabla F(3, 4, -5) =< 6, 8, 10 >$ and $\nabla G(3, 4, -5) =< 2, 3, 4 >$;
$\vec{n} = \nabla F(3, 4, -5) \times \nabla G(3, 4, -5) =< 2, -4, 2 >$.
Therefore, the equation of the plane normal to the ellipse is

$$< 2, -4, 2 > \cdot < x - 3, y - 4, z + 5 >= 0 \Rightarrow x - 2y + z + 10 = 0$$

53. sphere: $x^2 + y^2 + z^2 = r^2$ and cone: $z^2 = a^2x^2 + b^2y^2$.
Let $F(x, y, z) = x^2 + y^2 + z^2 - r^2$ and $G(x, y, z) = a^2x^2 + b^2y^2 - z^2$;
$\nabla F =< 2x, 2y, 2z >$ and $\nabla G =< 2a^2x, 2b^2y, -2z >$;
$\nabla F \cdot \nabla G = 4a^2x^2 + 4b^2y^2 - 4z^2 = 4(a^2x^2 + b^2y^2 - z^2) = 0$. Since the points (x, y, z) lies on the cone $z^2 = a^2x^2 + b^2y^2$.

54. circular ellipsoid $x^2 + y^2 + 2z^2 = 2$. Let $F(x, y, z) = x^2 + y^2 + z^2 - 2$; $\nabla F =< 2x, 2y, 4z >$. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be the points of the contact of the tangent plane $\mathbf{P}_1$ and $\mathbf{P}_2$ with the ellipsoid respectively.
Then the equation of the tangent plane $\mathbf{P}_1$ is

$$2x_1(x - x_1) + 2y_1(y - y_1) + 4z_1(z - z_1) = 0$$

And the equation of the tangent plane $\mathbf{P}_2$ is

$$2x_2(x - x_2) + 2y_2(y - y_2) + 4z_2(z - z_2) = 0$$

Set $x = y = 0$, for $\mathbf{P}_1$, we get

$$-2x_1^2 - 2y_1^2 + 4z_1(z - z_1) = 0 \Rightarrow z = \frac{2x_1^2 + 2y_1^2}{4z_1} + z_1 = \frac{2(2 - 2z_1^2)}{4z_1} + z_1 = \frac{1}{z_1}$$

and for $\mathbf{P}_2$, we get

$$-2x_2^2 - 2y_2^2 + 4z_2(z - z_2) = 0 \Rightarrow z = \frac{2x_2^2 + 2y_2^2}{4z_2} + z_2 = \frac{2(2 - 2z_2^2)}{4z_2} + z_2 = \frac{1}{z_2}$$

Therefore, if P_1 and P_2 have the same z-coordinate, then $\mathbf{P}_1$ and $\mathbf{P}_2$ intersect the same z-axis.

62. (a) If the two surfaces are tangent surfaces at P, then tangent vector of the two surfaces are parallel. Since $\nabla f(P)$ parallel to the tangent vector and $\nabla g(P)$ also parallel to the tangent vector, therefore, $\nabla f(P) \times \nabla g(P) = 0$.
If $\nabla f(P) \times \nabla g(P) = 0$, then the normal vector of two surfaces are parallel and so their tangent vector are also parallel. So the two surfaces are tangent at P.

(b) If the two surfaces are orthogonal at P, then the two normal vector $\nabla f(P)$ and $\nabla g(P)$ are orthogonal. Thus, $\nabla f(P) \cdot \nabla g(P) = 0$.
If $\nabla f(P) \cdot \nabla g(P) = 0$, then the two normal vector $\nabla f(P)$ and $\nabla g(P)$ are orthogonal. Thus, the two surfaces are orthogonal at P.

64. $f(x, y) = (\sqrt[3]{x} + \sqrt[3]{y})^3$.
Let $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} r(\sqrt[3]{\cos\theta} + \sqrt[3]{\sin\theta})^3 = 0 = f(0, 0)$$

Therefore, f is continuous at $(0, 0)$.
Let $\vec{u} =< u_1, u_2 >$, then

$$\begin{aligned} D_{\vec{u}} f(0,0) &= \lim_{h\to 0} \frac{f(0 + hu_1, 0 + hu_2) - f(0, 0)}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt[3]{hu_1} + \sqrt[3]{hu_2})^3}{h} \\ &= \lim_{h\to 0} \frac{h(\sqrt[3]{u_1} + \sqrt[3]{u_2})^3}{h} \\ &= (\sqrt[3]{u_1} + \sqrt[3]{u_2})^3 \end{aligned}$$

From the directional derivatives, we know $f_x(0,0) = 1$ and $f_y(0,0) = 1$.
Then

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - < f_x(0,0), f_y(0,0) > \cdot < h-0, k-0 >}{\sqrt{h^2+k^2}}$$
$$= \lim_{(h,k)\to(0,0)} \frac{(\sqrt[3]{h} + \sqrt[3]{k})^3 - (h+k)}{\sqrt{h^2+k^2}}$$

Take $h = r\cos\theta$ and $k = r\sin\theta$, then

$$\begin{aligned}\lim_{(h,k)\to(0,0)} \frac{(\sqrt[3]{h} + \sqrt[3]{k})^3 - (h+k)}{\sqrt{h^2+k^2}} &= \lim_{r\to 0} \frac{r(\sqrt[3]{\cos\theta} + \sqrt[3]{\sin\theta})^3 - r(\cos\theta + \sin\theta)}{r} \\ &= (\sqrt[3]{\cos\theta} + \sqrt[3]{\sin\theta})^3 - (\cos\theta + \sin)\end{aligned}$$

which has different values for different values of θ. Therefore it is not differentiable at the origin.

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