MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 8
Suggested Solution

4. $w = \ln(u + v + z);\ u = \cos^2 t,\ v = \sin^2 t,\ z = t^2.$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial v} = \frac{\partial w}{\partial z} = \frac{1}{u+v+z};\ \frac{\partial u}{\partial t} = -2\cos t \sin t,\ \frac{\partial v}{\partial t} = 2\sin t \cos t,\ \frac{\partial z}{\partial t} = 2t$$

Chain Rule:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t} \\
&= \left(\frac{1}{u+v+z}\right)(-2\cos t\sin t) + \left(\frac{1}{u+v+z}\right)(2\sin t\cos t) + \left(\frac{1}{u+v+z}\right)(2t) \\
&= \frac{2t}{u+v+z} \\
&= \frac{2t}{\cos^2 t + \sin^2 t + t^2} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

Expressing w explicitly as a function of t:

$w = \ln(u+v+z) = \ln(\cos^2 t + \sin^2 t + t^2) = \ln(1+t^2)$. Then, $\dfrac{\partial w}{\partial t} = \left(\dfrac{1}{1+t^2}\right)(2t) = \dfrac{2t}{1+t^2}$.

8. $w = yz + zx + xy;\ x = s^2 - t^2,\ y = s^2 + t^2,\ z = s^2t^2$

$\frac{\partial w}{\partial x} = z + y,\ \frac{\partial w}{\partial y} = z + x,\ \frac{\partial w}{\partial z} = y + x;$

$\frac{\partial x}{\partial s} = 2s,\ \frac{\partial y}{\partial s} = 2s,\ \frac{\partial z}{\partial s} = 2st^2;$

$\frac{\partial x}{\partial t} = -2t,\ \frac{\partial y}{\partial t} = 2t,\ \frac{\partial z}{\partial t} = 2s^2t;$

$$\begin{aligned}
\frac{\partial w}{\partial s} &= (z+y)(2s) + (z+x)(2s) + (y+x)(2st^2) \\
&= (s^2t^2 + s^2 + t^2)(2s) + (s^2t^2 + s^2 - t^2)(2s) + (s^2 + t^2 + s^2 - t^2)(2st^2) \\
&= 2s^3t^2 + 2s^3 + 2st^2 + 2s^3t^2 + 2s^3 - 2st^2 + 4s^3t^2 \\
&= 8s^3t^2 + 4s^3 \\
\frac{\partial w}{\partial t} &= (z+y)(-2t) + (z+x)(2t) + (y+x)(2s^2t) \\
&= (s^2t^2 + s^2 + t^2)(-2t) + (s^2t^2 + s^2 - t^2)(2t) + (s^2 + t^2 + s^2 - t^2)(2s^2t) \\
&= -2s^2t^3 - 2s^2t - 2t^3 + 2s^2t^3 + 2s^2t - 2t^3 + 4s^4t \\
&= 4s^4t - 4t^3
\end{aligned}$$

12. $r = \dfrac{p}{q} + \dfrac{q}{s} + \dfrac{s}{p};\ p = e^{yz},\ q = e^{xz},\ s = e^{xy}$

$\dfrac{\partial r}{\partial p} = \dfrac{1}{q} - \dfrac{s}{p^2},\ \dfrac{\partial r}{\partial q} = \dfrac{1}{s} - \dfrac{p}{q^2},\ \dfrac{\partial r}{\partial s} = \dfrac{1}{p} - \dfrac{q}{s^2};$

$\dfrac{\partial p}{\partial x} = 0,\ \dfrac{\partial p}{\partial y} = ze^{yz},\ \dfrac{\partial p}{\partial z} = ye^{yz};$

$$\frac{\partial q}{\partial x} = ze^{xz},\ \frac{\partial q}{\partial y} = 0,\ \frac{\partial q}{\partial z} = xe^{xz};$$
$$\frac{\partial s}{\partial x} = ye^{xy},\ \frac{\partial s}{\partial y} = xe^{xy},\ \frac{\partial s}{\partial z} = 0;$$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \left(\frac{1}{q} - \frac{s}{p^2}\right)(0) + \left(\frac{1}{s} - \frac{p}{q^2}\right)(ze^{xz}) + \left(\frac{1}{p} - \frac{q}{s^2}\right)(ye^{xy}) \\
&= \left(\frac{1}{e^{xy}} - \frac{e^{yz}}{e^{2xz}}\right)(ze^{xz}) + \left(\frac{1}{e^{yz}} - \frac{e^{xz}}{e^{2xy}}\right)(ye^{xy}) \\
&= ze^{x(z-y)} - ze^{z(y-x)} + ye^{y(x-z)} - ye^{x(z-y)} \\
\frac{\partial r}{\partial y} &= \left(\frac{1}{q} - \frac{s}{p^2}\right)(ze^{yz}) + \left(\frac{1}{s} - \frac{p}{q^2}\right)(0) + \left(\frac{1}{p} - \frac{q}{s^2}\right)(xe^{xy}) \\
&= \left(\frac{1}{e^{xz}} - \frac{e^{xy}}{e^{2yz}}\right)(ze^{yz}) + \left(\frac{1}{e^{yz}} - \frac{e^{xz}}{e^{2xy}}\right)(xe^{xy}) \\
&= ze^{z(y-x)} - ze^{y(x-z)} + xe^{y(x-z)} - xe^{x(z-y)} \\
\frac{\partial r}{\partial z} &= \left(\frac{1}{q} - \frac{s}{p^2}\right)(ye^{yz}) + \left(\frac{1}{s} - \frac{p}{q^2}\right)(xe^{xz}) + \left(\frac{1}{p} - \frac{q}{s^2}\right)(0) \\
&= \left(\frac{1}{e^{xz}} - \frac{e^{xy}}{e^{2yz}}\right)(ye^{yz}) + \left(\frac{1}{e^{xy}} - \frac{e^{yz}}{e^{2xz}}\right)(xe^{xz}) \\
&= ye^{z(y-x)} - ye^{y(x-z)} + xe^{x(z-y)} - xe^{z(y-x)}
\end{aligned}$$

16. $p = f(v, w);\ v = v(x, y, z, t),\ w = w(x, y, z, t)$

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} \\
\frac{\partial p}{\partial y} &= \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} \\
\frac{\partial p}{\partial z} &= \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z} \\
\frac{\partial p}{\partial t} &= \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial t}
\end{aligned}$$

24. $xyz = \sin(x + y + z)$. Let $F(x, y, z) = \sin(x + y + z) - xyz$. Then

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x(x, y, z)}{F_z(x, y, z)} \\
&= -\frac{\cos(x + y + z) - yz}{\sin(x + y + z) - xy} \\
&= \frac{yz - \cos(x + y + z)}{\cos(x + y + z) - xy}
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= -\frac{F_y(x,y,z)}{F_z(x,y,z)} \\
&= -\frac{\cos(x+y+z)-xz}{\sin(x+y+z)-xy} \\
&= \frac{xz-\cos(x+y+z)}{\cos(x+y+z)-xy}
\end{aligned}
$$

27. $w = xy\ln(u+v);\ u = (x^2+y^2)^{1/3},\ v = (x^3+y^3)^{1/2}$

$\frac{\partial w}{\partial u} = \frac{xy}{u+v},\ \frac{\partial w}{\partial v} = \frac{xy}{u+v};$

$\frac{\partial u}{\partial x} = \frac{2x}{3}(x^2+y^2)^{-2/3},\ \frac{\partial u}{\partial y} = \frac{2y}{3}(x^2+y^2)^{-2/3};$

$\frac{\partial v}{\partial x} = \frac{3x^2}{2}(x^3+y^3)^{-1/2},\ \frac{\partial v}{\partial y} = \frac{3y^2}{2}(x^3+y^3)^{-1/2};$

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial x} \\
&= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \\
&= \left(\frac{xy}{u+v}\right)\left(\frac{2x}{3}(x^2+y^2)^{-2/3}\right) + \left(\frac{xy}{u+v}\right)\left(\frac{3x^2}{2}(x^3+y^3)^{-1/2}\right) + y\ln(u+v) \\
&= \frac{2x^2y}{3(x^2+y^2)^{2/3}[(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}]} \\
&\quad + \frac{3x^3y}{2(x^3+y^3)^{1/2}[(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}]} \\
&\quad + y\ln((x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}) \\
\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial y} \\
&= \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \\
&= \left(\frac{xy}{u+v}\right)\left(\frac{2y}{3}(x^2+y^2)^{-2/3}\right) + \left(\frac{xy}{u+v}\right)\left(\frac{3y^2}{2}(x^3+y^3)^{-1/2}\right) + x\ln(u+v) \\
&= \frac{2xy^2}{3(x^2+y^2)^{2/3}[(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}]} \\
&\quad + \frac{3xy^3}{2(x^3+y^3)^{1/2}[(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}]} \\
&\quad + x\ln((x^2+y^2)^{1/3}+(x^3+y^3)^{1/2})
\end{aligned}
$$

32. $z^3 + (x+y)z^2 + x^2 + y^2 = 13;\ P(2,2,1)$

$$\begin{aligned}
\frac{\partial}{\partial x}(z^3 + (x+y)z^2 + x^2 + y^2) &= \frac{\partial}{\partial x}(13) \\
3z^2 \cdot \frac{\partial z}{\partial x} + z^2 + 2(x+y)z \cdot \frac{\partial z}{\partial x} + 2x &= 0 \\
\frac{\partial z}{\partial x} &= \frac{-2x - z^2}{3z^2 + 2xz + 2yz}
\end{aligned}$$

At $P(2,2,1)$, $\dfrac{\partial z}{\partial x} = -\dfrac{5}{11}$.

$$\begin{aligned}
\frac{\partial}{\partial y}(z^3 + (x+y)z^2 + x^2 + y^2) &= \frac{\partial}{\partial y}(13) \\
3z^2 \cdot \frac{\partial z}{\partial y} + z^2 + 2(x+y)z \cdot \frac{\partial z}{\partial y} + 2y &= 0 \\
\frac{\partial z}{\partial x} &= \frac{-2y - z^2}{3z^2 + 2xz + 2yz}
\end{aligned}$$

At $P(2,2,1)$, $\dfrac{\partial z}{\partial x} = -\dfrac{5}{11}$.
Therefore, the tangent plane at P to the surface is

$$z - 1 = \left(-\frac{5}{11}\right)(x-2) + \left(-\frac{5}{11}\right)(y-2) \Rightarrow 5x + 5y + 11z = 31$$

40. $w = f(x,y);\ x = r\cos\theta,\ y = r\sin\theta;$
$\dfrac{\partial x}{\partial r} = \cos\theta,\ \dfrac{\partial x}{\partial \theta} = -r\sin\theta;\ \dfrac{\partial y}{\partial r} = \sin\theta,\ \dfrac{\partial y}{\partial \theta} = r\cos\theta;$

Then

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} \\
&= \left(\frac{\partial w}{\partial x}\right)(\cos\theta) + \left(\frac{\partial w}{\partial y}\right)(\sin\theta) \\
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta} \\
&= \left(\frac{\partial w}{\partial x}\right)(-r\sin\theta) + \left(\frac{\partial w}{\partial y}\right)(r\cos\theta)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \\
= &\left(\left(\frac{\partial w}{\partial x}\right)(\cos\theta) + \left(\frac{\partial w}{\partial y}\right)(\sin\theta)\right)^2 + \frac{1}{r^2}\left(\left(\frac{\partial w}{\partial x}\right)(-r\sin\theta) + \left(\frac{\partial w}{\partial y}\right)(r\cos\theta)\right)^2 \\
= &\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2
\end{aligned}$$

45. $w = f(x,y);\ x = r\cos\theta,\ y = r\sin\theta;$
$\dfrac{\partial x}{\partial r} = \cos\theta,\ \dfrac{\partial x}{\partial \theta} = -r\sin\theta;\dfrac{\partial y}{\partial r} = \sin\theta,\ \dfrac{\partial y}{\partial \theta} = r\cos\theta;$

Then

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \\
&= \cos\theta\frac{\partial w}{\partial x} + \sin\theta\frac{\partial w}{\partial y} \\
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} \\
&= -r\sin\theta\frac{\partial w}{\partial x} + r\cos\theta\frac{\partial w}{\partial y} \\
\frac{\partial^2 w}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) \\
&= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial w}{\partial x} + \sin\theta\frac{\partial w}{\partial y}\right) \\
&= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial y}\right) \\
&= \cos\theta\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 w}{\partial y\partial x}\cdot\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 w}{\partial^2 y}\cdot\frac{\partial y}{\partial r}\right) \\
&= \cos\theta\left(\frac{\partial^2 w}{\partial x^2}\cdot\cos\theta + \frac{\partial^2 w}{\partial y\partial x}\cdot\sin\theta\right) + \sin\theta\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\cos\theta + \frac{\partial^2 w}{\partial^2 y}\cdot\sin\theta\right) \\
&= \cos^2\frac{\partial^2 w}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 w}{\partial^2 y} \\
\frac{\partial^2 w}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial \theta}\right) \\
&= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial w}{\partial x} + r\cos\theta\frac{\partial w}{\partial y}\right) \\
&= -r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial x}\right) - r\cos\theta\frac{\partial w}{\partial x} \\
&\quad +r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial y}\right) - r\sin\theta\frac{\partial w}{\partial y} \\
&= -r\sin\theta\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 w}{\partial y\partial x}\cdot\frac{\partial y}{\partial \theta}\right) - r\cos\theta\frac{\partial w}{\partial x} \\
&\quad +r\cos\theta\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 w}{\partial^2 y}\cdot\frac{\partial y}{\partial \theta}\right) - r\sin\theta\frac{\partial w}{\partial y} \\
&= -r\sin\theta\left(\frac{\partial^2 w}{\partial x^2}\cdot(-r\sin\theta) + \frac{\partial^2 w}{\partial y\partial x}\cdot(r\cos\theta)\right) - r\cos\theta\frac{\partial w}{\partial x} \\
&\quad +r\cos\theta\left(\frac{\partial^2 w}{\partial x\partial y}\cdot(-r\sin\theta) + \frac{\partial^2 w}{\partial^2 y}\cdot(r\cos\theta)\right) - r\sin\theta\frac{\partial w}{\partial y} \\
&= r^2\sin^2\frac{\partial^2 w}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 w}{\partial^2 y} - r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial w}{\partial y}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \\
= & \left(\cos^2 \frac{\partial^2 w}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 w}{\partial^2 y}\right) \\
& + \frac{1}{r}\left(\cos\theta\frac{\partial w}{\partial x} + \sin\theta\frac{\partial w}{\partial y}\right) \\
& + \frac{1}{r^2}\left(r^2\sin^2\frac{\partial^2 w}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x \partial y} + r^2\cos^2\theta\frac{\partial^2 w}{\partial^2 y} - r\cos\theta\frac{\partial w}{\partial x} - r\sin\theta\frac{\partial w}{\partial y}\right) \\
= & \left(\cos^2 \frac{\partial^2 w}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 w}{\partial^2 y}\right) \\
& + \left(\frac{1}{r}\cos\theta\frac{\partial w}{\partial x} + \frac{1}{r}\sin\theta\frac{\partial w}{\partial y}\right) \\
& + \left(\sin^2\frac{\partial^2 w}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 w}{\partial^2 y} - \frac{1}{r}\cos\theta\frac{\partial w}{\partial x} - \frac{1}{r}\sin\theta\frac{\partial w}{\partial y}\right) \\
= & \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}
\end{aligned}
$$

46. $w = \frac{1}{r} f\left(t - \frac{r}{a}\right)$; $r = \sqrt{x^2 + y^2 + z^2}$;
$\frac{\partial r}{\partial x} = x(x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r};$
Let $u = t - \frac{r}{a}$, then $\frac{\partial u}{\partial t} = 1,\ \frac{\partial u}{\partial r} = -\frac{1}{a}$ and $w = \frac{1}{r} f(u)$
Therefore,

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{r}\right) f(u) + \frac{1}{r}\frac{\partial}{\partial x}(f(u)) \\
&= \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \cdot \frac{\partial r}{\partial x} \cdot f(u) + \frac{1}{r}\frac{\partial}{\partial u}(f(u)) \cdot \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \\
&= -\frac{1}{r^2}\left(\frac{x}{r}\right) f(u) + \frac{1}{r}(f'(u))\left(-\frac{1}{a}\right)\left(\frac{x}{r}\right) \\
&= \frac{x}{r^2}\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) \\
\frac{\partial^2 w}{\partial^2 x} &= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) + \left(\frac{x}{r^2}\right)\frac{\partial}{\partial x}\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) \\
&= \left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right)\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) + \frac{x}{r^2}\left(\frac{x f(u)}{r^3} + \frac{x f'(u)}{r^2 a} + \frac{x f''(u)}{r a^2}\right)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial^2 w}{\partial^2 y} &= \left(\frac{1}{r^2} - \frac{2y^2}{r^4}\right)\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) + \frac{y}{r^2}\left(\frac{y f(u)}{r^3} + \frac{y f'(u)}{r^2 a} + \frac{y f''(u)}{r a^2}\right) \\
\frac{\partial^2 w}{\partial^2 z} &= \left(\frac{1}{r^2} - \frac{2z^2}{r^4}\right)\left(-\frac{1}{r} f(u) - \frac{f'(u)}{a}\right) + \frac{z}{r^2}\left(\frac{z f(u)}{r^3} + \frac{z f'(u)}{r^2 a} + \frac{z f''(u)}{r a^2}\right)
\end{aligned}
$$

Add them together, we get

$$\begin{aligned}\frac{\partial^2 w}{\partial^2 x}+\frac{\partial^2 w}{\partial^2 y}+\frac{\partial^2 w}{\partial^2 z} &= \frac{1}{r^2}\left(-\frac{1}{r}f(u)-\frac{f'(u)}{a}\right)+\left(\frac{f(u)}{r^3}+\frac{f'(u)}{r^2 a}+\frac{f''(u)}{ra^2}\right)\\ &= \frac{f''(u)}{ra^2}\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{1}{r}\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t}\\ &= \frac{f'(u)}{r}\\ \frac{\partial^2 w}{\partial t^2} &= \frac{1}{r}\frac{\partial f'(u)}{\partial u}\cdot\frac{\partial u}{\partial t}\\ &= \frac{f''(u)}{r}\end{aligned}$$

Therefore,

$$\frac{\partial^2 w}{\partial^2 x}+\frac{\partial^2 w}{\partial^2 y}+\frac{\partial^2 w}{\partial^2 z} = \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}$$

51. $w=f(x,y); x=u\cos\alpha-v\sin\alpha,\ y=u\sin\alpha+v\cos\alpha$ where α is a constant.
$\frac{\partial x}{\partial u}=\cos\alpha,\ \frac{\partial x}{\partial v}=-\sin\alpha;\ \frac{\partial y}{\partial u}=\sin\alpha,\ \frac{\partial y}{\partial v}=\cos\alpha;$

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= \cos\alpha\frac{\partial w}{\partial x}+\sin\alpha\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}\\ &= -\sin\alpha\frac{\partial w}{\partial x}+\cos\alpha\frac{\partial w}{\partial y}\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial w}{\partial u}\right)^2+\left(\frac{\partial w}{\partial v}\right)^2 &= \left(\cos\alpha\frac{\partial w}{\partial x}+\sin\alpha\frac{\partial w}{\partial y}\right)^2+\left(-\sin\alpha\frac{\partial w}{\partial x}+\cos\alpha\frac{\partial w}{\partial y}\right)^2\\ &= \left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\end{aligned}$$

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