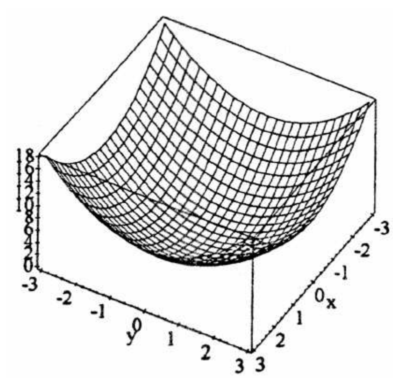
MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 3
Suggested Solution

Exercises 12.6

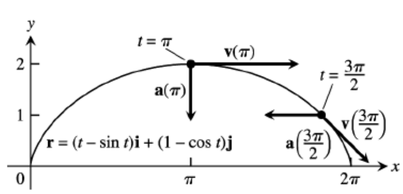
4. g, cone
5. l, hyperbolic paraboloid
6. e, paraboloid
7. b, cylinder
8. j, hyperboloid
51. $z = x^2 + y^2, \quad -3 \le x \le 3, \quad -3 \le y \le 3$



Exercises 13.1

1. $x = t + 1$ and $y = t^2 - 1 \Rightarrow y = (x-1)^2 - 1 = x^2 - 2x$;
$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{a} = \frac{dv}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{j}$ at time $t = 1$.

7. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$.
Therefore, for $t = \pi$, $\mathbf{v}(\pi) = 2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$.
And for $t = \frac{3\pi}{2}$, $\mathbf{v}(\frac{3\pi}{2}) = \mathbf{i} - \mathbf{j}$ and $\mathbf{a}(\frac{3\pi}{2}) = -\mathbf{i}$.



11. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$;
Speed: $\left|\mathbf{v}(\frac{\pi}{2})\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(3\cos\frac{\pi}{2}\right)^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}(\frac{\pi}{2})}{|\mathbf{v}(\frac{\pi}{2})|}$

19. $\mathbf{r} = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}$;
At $t = 0 \Rightarrow \mathbf{v}(t) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, -1, 1) \Rightarrow x = 0 + t = t, y = -1$ and $z = 1 + t$ are parametric equations of the tangent line

23. (d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$
   (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
   (iii) clockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$
   (e) $\mathbf{v}(t) = -(2t\sin t)\mathbf{i} + (2t\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2\sin t + 2t\cos t)\mathbf{i} + (2\cos t - 2t\sin t)\mathbf{j}$
   (i) $|\mathbf{v}(t)| = \sqrt{(-(2t\sin t)^2 + (2t\cos t)^2} = \sqrt{4t^2(\sin^2 t + \cos^2 t)} = 2|t| = 2t$ for $t \geq 0 \Rightarrow$ variable speed;
   (ii) $\mathbf{v} \cdot \mathbf{a} = 4(\sin^2 t + t^2 \sin t \cos t) + 4(t\cos^2 t - t^2 \cos t \sin t) = 4t \neq 0$ in general $\Rightarrow$ not orthogonal in general;
   (iii) counterclockwise movement;
   (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

24. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1)$, the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

27. $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$
Therefore, $\mathbf{r} \cdot \mathbf{r}$ is a constant $\Rightarrow |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ is a constant.

Exercises 13.2

1. $\int_0^1 t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}\, dt = \left[t^3\right]_0^1 \mathbf{i} + [7]_0^1 \mathbf{j} + [(t+1)]_0^1 \mathbf{k} = \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}$

17. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at

time $t = 0$ it has speed 2

$$\begin{aligned}
&\Rightarrow\quad \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}3\mathbf{i} - \mathbf{j} + \mathbf{k} \\
&\Rightarrow\quad \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k} \\
&\Rightarrow\quad \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2 \\
&\because\quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2 \\
&\Rightarrow\quad \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k} \\
&\Rightarrow\quad \mathbf{r}(t) = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})
\end{aligned}$$

Exercises 13.3

1. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0$; $\mathbf{a} = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \sqrt{a^2} = |a| \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\mathbf{N} = |a|\mathbf{N}$

6. $\mathbf{r} = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow |\mathbf{v}| = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2$; $\mathbf{a} = (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{i} + (e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} = (-2e^t\sin t)\mathbf{i} + (2e^t\cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{6} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

15. $\mathbf{r} = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow \mathbf{v} = (\sqrt{2})\mathbf{i} + (\sqrt{2})\mathbf{j} + (-2t)\mathbf{k} \Rightarrow |\mathbf{v}| = 2\sqrt{1+t^2}$.
$\therefore$ Length$= \int_0^1 2\sqrt{1+t^2}\,dt = \left[2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right)\right]_0^1 = \sqrt{2} + \ln(1+\sqrt{2})$

18. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + (4t)\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + (4)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2} = 4\sqrt{2}$. $\therefore$ Length$= \int_0^{\pi/2} 4\sqrt{2}\,dt = 2\pi\sqrt{2}$.
    (c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{2}$. $\therefore$ Length$= \int_{-2\pi}^{0} \sqrt{2}\,dt = 2\pi\sqrt{2}$.

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