Homework 6 solutions

April 27, 2020

Problem 1. Since $\Delta = d\delta + \delta d$, using $d^2 = 0$, we have

$$\Delta d = (d\delta + \delta d)d = d\delta d + \delta d^2 = d\delta d = d^2\delta + d\delta d = d(d\delta + \delta d) = d\Delta.$$

Similar, using $\delta^2 = 0$, we have $\Delta\delta = \delta\Delta$. Finally, using $\delta = (-1)^{mp+m+1} * d*$ and $*^2 = (-1)^{p(m-p)}$ on $\Omega^p(M)$ where $m = \dim(M)$, we have on $\Omega^p(M)$,

$$\Delta* = (d\delta + \delta d)* = (-1)^{p+1} d * d + (-1)^{m(p+1)+1} * d * d* = *(d\delta + \delta d) = *\Delta.$$

Problem 2. Let $\alpha := V^\flat \in \Omega^1(\Sigma)$ be the dual 1-form of V. By Hodge decomposition, there exist a harmonic 1-form $\alpha_H \in \Omega^1(\Sigma)$, $f \in C^\infty(\Sigma)$ and $\omega \in \Omega^2(\Sigma)$ such that

$$\alpha = \alpha_H + df + \delta\omega.$$

Taking its dual again, we have

$$V = V_H + \nabla f + (\delta\omega)^\sharp.$$

Since α_H is harmonic, $d\alpha = 0$ and $\delta\alpha = 0$. By Poincare lemme, any closed 1-form is exact, so locally $\alpha = dh$ for some function h, which is harmonic as $0 = \delta\alpha = \delta dh = \Delta h$. It remains to show that $(\delta\omega)^\sharp = J(\nabla g)$ for some function g. Since Σ is two-dimensional, we can write $\omega = -g dV$ where dV is the volume form for the oriented surface (Σ, g). Then, $\delta\omega = *(dg)$. Let $\{e_1, e_2\}$ be a positive local orthonormal basis of T^*M, we have $*e_1 = e_2$ and $*e_2 = -e_1$, i.e. $*$ is the same as J under the identification between TM and T^*M. Therefore, $(\delta\omega)^\sharp = J(\nabla g)$.

Problem 3. By Hodge decomposition, for any $\alpha \in [\alpha_0]$, we have a unique decomposition $\alpha = \alpha_H + d\eta + \delta\beta$ for some $\eta \in \Omega^{p-1}$, $\beta \in \Omega^{p+1}$. Since α is closed,

$$0 = d\alpha = d\alpha_H + d^2\eta + d\delta\beta = d\delta\beta.$$

Moreover, $\delta(\delta\beta) = 0$. So $\delta\beta$ is a harmonic form, hence must vanish by the L^2-orthogonality in the decomposition. Using $\alpha = \alpha_H + d\eta$, we compute

$$E(\alpha) = \|\alpha_H\|^2_{L^2} + \|d\eta\|^2_{L^2} + 2\langle\alpha_H, d\eta\rangle_{L^2} = \|\alpha_H\|^2_{L^2} + \|d\eta\|^2_{L^2} \geq \|\alpha_H\|^2_{L^2}.$$

Note that we have used $\langle\alpha_H, d\eta\rangle_{L^2} = \langle\delta\alpha_H, \eta\rangle_{L^2} = 0$. The inequality above implies the assertion.

Problem 4. Since $S\alpha(x, y) = (D_x\alpha)(y) + (D_y\alpha)(x)$, we have $\langle S\alpha, h\rangle = 2\langle D\alpha, h\rangle = 2\langle\alpha, \delta h\rangle$ for any $h \in \Gamma(S^2M)$. Taking divergence and using Ricci identity, we have

$$\begin{aligned} \delta(S\alpha)(Y) &= -\textstyle\sum_{i=1}^m (D_{e_i}D_{e_i}\alpha(Y) + D_{e_i}D_Y\alpha(e_i)) \\ &= D^*D\alpha(Y) - \textstyle\sum_{i=1}^m (D_Y D_{e_i}\alpha(e_i) + R(e_i, Y)e_i) \end{aligned}$$

which implies the desired identity in (b). Finally, if $\alpha^\sharp$ is a Killing vector field, then $\delta\alpha = 0$, therefore

$$0 = \langle\delta S\alpha, \alpha\rangle = \langle D^*D\alpha, \alpha\rangle - \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

Integrating over M and using $\mathrm{Ric} < 0$, we conclude that $\alpha = 0$.

Problem 5. Let $f \in C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_M f = 1 \text{ in } M, \\ f|_{\partial M} = 0. \end{cases}$$

From Divergence Theorem, we have

$$\mathrm{Vol}(M) = \int_M \Delta_M f = \int_{\partial M} \frac{\partial f}{\partial \nu}.$$

By Schwarz inequality, $(\Delta_M f)^2 \leq m|\mathrm{Hess}_M f|^2$. Plug this into Reilly's formula and using $\mathrm{Ric} \geq 0$, we have

$$\frac{m-1}{m}\mathrm{Vol}(M) \geq \int_{\partial M} H \left(\frac{\partial f}{\partial \nu}\right)^2.$$

Combining the two inequalities above and apply Schwarz inequality,

$$\mathrm{Vol}(M)^2 = \left(\int_{\partial M} \frac{\partial f}{\partial \nu}\right)^2 \leq \left(\int_{\partial M} H \left(\frac{\partial f}{\partial \nu}\right)^2\right)\left(\int_{\partial M} \frac{1}{H}\right) \leq \frac{m-1}{m}\mathrm{Vol}(M)\int_{\partial M}\frac{1}{H}$$

which proves the desired inequality. If equality holds, then $\mathrm{Hess}_M f = \frac{1}{m} g$, which is parallel. Using the Ricci identity, $R(X,Y)\nabla f = 0$ and we can deduce that

$$\nabla f = \frac{1}{m} r \frac{\partial}{\partial r}$$

where $r = d_M(p, \cdot)$ is the Riemannian distance function of (M, g) from a point $p \in M$. This, in turn, implies that (M, g) is isometric to a Euclidean ball.