Lecture 11

Lemma 6.4. *Let $n, l \in \mathbb{N}, l < n$ and $\mu \in \mathscr{M}(X)$. Let $\xi = \{A_1, \cdots, A_k\}$ be a Borel partition of X. Then*

$$\frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\xi) \leq \frac{1}{l} H_{\mu_n}(\bigvee_{i=0}^{l-1} T^{-i}\xi) + \frac{2l \log k}{n},$$

where $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ T^{-i}$.

Proof. Fix $l < n$. For $j = 0, 1, \cdots, l-1$, define t_j to be the largest integer so that $t_j l + j \leq n$, i.e. $t_j = [\frac{n-j}{l}]$. Write $\{0, 1, \cdots, n-1\} = \{j, j+1, \cdots, t_j l + j - 1\} \cup S_j$ as a disjoint union, notice that $\sharp S_j \leq 2l$. Hence

$$\begin{aligned}
\bigvee_{i=0}^{n-1} T^{-i}\xi &= (\bigvee_{i=j}^{t_j l + j - 1} T^{-i}\xi) \bigvee (\bigvee_{i \in S_j} T^{-i}\xi) \\
&= [\bigvee_{r=0}^{t_j - 1} T^{-rl-j} (\bigvee_{i=0}^{l-1} T^{-i}\xi)] \bigvee (\bigvee_{i \in S_j} T^{-i}\xi),
\end{aligned}$$

therefore

$$\begin{aligned}
H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\xi) &\leq \sum_{r=0}^{t_j-1} H_\mu\big(T^{-rl-j}(\bigvee_{i=0}^{l-1} T^{-i}\xi)\big) + \sum_{i \in S_j} H_\mu(T^j \xi) \\
&\leq \sum_{r=0}^{t_j-1} H_\mu\big(T^{-rl-j}(\bigvee_{i=0}^{l-1} T^{-i}\xi)\big) + 2l \log k \\
&= \sum_{r=0}^{t_j-1} H_{\mu \circ T^{-rl-j}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + 2l \log k.
\end{aligned}$$

Summing over $j = 0, 1, \cdots, n-1$,

$$\begin{aligned}
l H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\xi) &\leq \sum_{j=0}^{l-1} \sum_{r=0}^{t_j-1} H_{\mu \circ T^{-rl-j}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + 2l^2 \log k \\
&= \sum_{m=0}^{n-l} H_{\mu \circ T^{-m}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + 2l^2 \log k \\
&\leq \sum_{m=0}^{n-1} H_{\mu \circ T^{-m}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + 2l^2 \log k.
\end{aligned}$$

Dividing by n on both sides,

$$\begin{aligned}\frac{l}{n}H_\mu(\bigvee_{i=0}^{n-1}T^{-i}\xi) &\le \frac{1}{n}\sum_{m=0}^{n-1}H_{\mu\circ T^{-m}}(\bigvee_{i=0}^{l-1}T^{-i}\xi)+\frac{2l^2\log k}{n}\\ &\le H_{\frac{1}{n}\sum_{m=0}^{n-1}\mu\circ T^{-m}}(\bigvee_{i=0}^{l-1}T^{-i}\xi)+\frac{2l^2\log k}{n}\\ &= H_{\mu_n}(\bigvee_{i=0}^{l-1}T^{-i}\xi)+\frac{2l^2\log k}{n},\end{aligned}$$

dividing by l, we complete the proof. □

Now we can prove the variational principle.

Proof of Theorem 6.2. Step 1. We first prove that $h_{top}(T) \ge h_\mu(T)$ for any $\mu \in M(X,T)$.

Fix $\mu \in M(X,T)$. Let $\xi = \{A_1, \cdots, A_k\}$ be a Borel partition of X. For any $\delta > 0$, pick compact $B_i \subseteq A_i$ with $\mu(A_i \setminus B_i) < \delta$, $i = 1, 2, \cdots, k$. Let $B_0 = X \setminus \cup_{i=1}^k B_i$, then $\mu(B_0) \le k\delta$. Moreover, $\eta := \{B_1, B_2, \cdots, B_k, B_0\}$ is a Borel partition of X, and $\beta := \{B_1 \cup B_0, B_2 \cup B_0, \cdots, B_k \cup B_0\}$ is an open cover of X. Since $\frac{\mu(A_i\cap B_j)}{\mu(B_j)} = 0$ or 1 if $1 \le j \le k$,

$$\begin{aligned}H_\mu(\xi|\eta) &= \sum_{j=0}^{k}\mu(B_j)\sum_{i=1}^{k}\phi(\frac{\mu(A_i\cap B_j)}{\mu(B_j)})\\ &= \mu(B_0)\sum_{i=1}^{k}\phi(\frac{\mu(A_i\cap B_0)}{\mu(B_0)}) \le k^2\delta.\end{aligned}$$

We claim that any member in $\bigvee_{i=0}^{n-1}T^{-i}\beta$ intersects at most 2^n many members of $\bigvee_{i=0}^{n-1}T^{-i}\eta$. To see this, if $\bigcap_{i=0}^{n-1}T^{-i}(B_0 \cup B_{t_i})$ intersects $\bigcap_{i=0}^{n-1}T^{-i}B_{s_i}$, then $s_i = 0$ or $s_i = t_i$, then claim follows. As a consequence,

$$N(\bigvee_{i=0}^{n-1}T^{-i}\beta) \ge \frac{1}{2^n}N(\bigvee_{i=0}^{n-1}T^{-i}\eta),$$

hence

$$\begin{aligned}\log N(\bigvee_{i=0}^{n-1}T^{-i}\beta) &\ge \log N(\bigvee_{i=0}^{n-1}T^{-i}\eta) - n\log 2\\ &\ge H_\mu(\bigvee_{i=0}^{n-1}T^{-i}\eta) - n\log 2,\end{aligned}$$

dividing by n and letting $n \to \infty$,

$$\begin{aligned} h_{top}(T,\beta) &\geq h_\mu(T,\eta) - \log 2 \\ &\geq h_\mu(T,\xi) - H_\mu(\xi|\eta) - \log 2 \\ &\geq h_\mu(T,\xi) - k^2\delta - \log 2. \end{aligned}$$

Hence

$$h_{top}(T) \geq h_\mu(T,\xi) - k^2\delta - \log 2,$$

letting $\delta \to 0$, we have

$$h_{top}(T) \geq h_\mu(T,\xi) - \log 2,$$

taking supremum over ξ,

$$h_{top}(T) \geq h_\mu(T) - \log 2.$$

Since μ is T-invariant, μ is also T^n-invariant, applying the above inequality to T^n, $h_{top}(T^n) \geq h_\mu(T^n) - \log 2$, hence $nh_{top}(T) \geq nh_\mu(T) - \log 2$, dividing by n and letting $n \to \infty$, we have $h_{top}(T) \geq h_\mu(T)$.

Step 2. We show that for any $\epsilon > 0$, there exists $\mu \in M(X,T)$, such that

$$h_\mu(T) \geq S(\epsilon),$$

where $S(\epsilon) := \overline{\lim}_{n\to\infty} \frac{1}{n} \log S_n(\epsilon)$, $S_n(\epsilon) := \sup\{\sharp E : E \subset X \text{ is } (n,\epsilon)\text{-separated }\}$.

Fix $\epsilon > 0$. For each $n \in \mathbb{N}$, pick $E_n \subset X$ which is (n,ϵ)-separated and $\sharp E = S_n(\epsilon)$. Define $\sigma_n = \frac{1}{S_n(\epsilon)} \sum_{x\in E_n} \delta_x$, where δ_x denotes the atomic measure. Define $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \sigma_n \circ T^{-i}$, clearly $\mu_n \in \mathscr{M}(X)$. By compactness, we can find a subsequence (n_j) of positive integers such that $\frac{1}{n_j} \log S_{n_j}(\epsilon) \to S(\epsilon)$ as $j \to \infty$ and $\mu_{n_j} \to \mu$ is the weak-$*$ topology, then $\mu \in M(X,T)$.

Next we show $h_\mu(T) \geq S(\epsilon)$. By Lemma 6.3, we find a partition $\xi = \{A_1, \cdots, A_k\}$ of X such that $\text{diam}(\xi) < \epsilon$ and $\mu(\partial A_i) = 0$ for each i. Observe that any member of $\bigvee_{i=0}^{n-1} T^{-i}\xi$ contains at most one point in E_n. To see this, suppose $x, y \in E_n$ and $x, y \in \bigcap_{i=0}^{n-1} T^{-i} A_{t_i}$, then $T^i x, T^i y \in A_{t_i} \in \xi$ hence $d(T^i x, T^i y) < \epsilon$ for $i = 0, 1, \cdots, n-1$, namely $d_n(x,y) < \epsilon$, a contradiction. Consequently,

$$H_{\sigma_n}\big(\bigvee_{i=0}^{n-1} T^{-i}\xi\big) = \log S_n(\epsilon).$$

Applying Lemma 6.4 to σ_n, we have for any $l \leq n$,

$$\frac{1}{n}\log S_n(\epsilon) = \frac{1}{n} H_{\sigma_n}\big(\bigvee_{i=0}^{n-1} T^{-i}\xi\big) \leq \frac{1}{l} H_{\mu_n}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + \frac{2l\log k}{n}.$$

Fix l,

$$\frac{1}{n_j}\log S_{n_j}(\epsilon) \leq \frac{1}{l} H_{\mu_{n_j}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) + \frac{2l\log k}{n_j},$$

letting $j \to \infty$,

$$S(\epsilon) \le \frac{1}{l} \lim_{j\to\infty} H_{\mu_{n_j}}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big) = \frac{1}{l} H_{\mu}\big(\bigvee_{i=0}^{l-1} T^{-i}\xi\big),$$

letting $l \to \infty$, we complete the proof.

We have to explain the last "=". Recall that if $\mu_n \to \infty$ in the weak-$*$ topology, then

(i) $\overline{\lim}_{n\to\infty} \mu_n(K) \le \mu(K)$ if K is compact.

(ii) $\underline{\lim}_{n\to\infty} \mu_n(V) \ge \mu(V)$ if V is open.

(iii) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ if $\mu(\partial A) = 0$.

For completeness, we give a proof of these facts. First (iii) follows from (i) and (ii), since

$$\overline{\lim_{n\to\infty}} \mu_n(A) \le \overline{\lim_{n\to\infty}} \mu_n(\bar{A}) \le \mu(\bar{A}) = \mu(A) = \mu(\mathring{A}) \le \underline{\lim}_{n\to\infty} \mu_n(\mathring{A}) \le \underline{\lim}_{n\to\infty} \mu_n(A).$$

For (i), suppose K is compact. For any $\epsilon > 0$, there exists U open, such that $U \supset K$ and $\mu(U \setminus K) < \epsilon$. By Urysohn's lemma, there is $f \in C(X)$, such that $0 \le f \le 1$, $f|_K = 1$ and $f|_{U^c} = 0$, then

$$\mu_n(K) \le \int_X f d\mu_n \to \int_X f d\mu \le \mu(U) < \mu(K) + \epsilon,$$

hence $\overline{\lim}_{n\to\infty} \mu_n(K) \le \mu(K) + \epsilon$, (i) is proved. A similar argument proves (ii). Recall $\mu(\partial A_i) = 0$ for $A_i \in \xi$, then $\mu\big(\partial \bigcap_{i=0}^{n-1} A_{t_i}\big) \le \mu\big(\bigcup_i T^{-i}\partial A_{t_i}\big) = 0$, by (iii) the proof is completed.

□