2017-18 MATH1010
Lecture 8: Differentiation
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1 Motivation

1.1 Velocity

Given a function $f(x)$, it is useful to know how $f(x)$ responds to small changes of x. For example, let t be the time used and $f(t)$ be the distance of an object travelled, then the rate of changes is the velocity.

Example 1 $f(t) = 2t$, then the velocity is $2t/t = 2$.

However, the method does not work when $f(t)$ is not linear.

Example 2 $f(t) = t^2$, find the velocity at $t = 1$. We can use approximation:

$$\text{velocity} = \frac{\text{change in distance}}{\text{change in time}}.$$

Suppose the time used is h, then the distance traveled is $f(1+h) - f(1)$. Hence the approximation is given by

$$\frac{f(1+h) - f(1)}{h}.$$

Of course the smaller the h, the better the approximation:

h	0.1	0.01	0.001	0.0001
$\frac{f(1+h)-f(1)}{h}$	2.1	2.01	2.001	2.0001

The velocity is given by $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = 2$.

Example 3 $f(t) = t^2$, find the velocity at $t = 1$ by limit.
Answer:

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + 2h}{h} = \lim_{h\to 0}(h+2) = 2.$$

Example 4 Given $f(t) = t^2$, find the velocity at time $t = t_0$.
Answer: Although t_0 looks like a variable, we assume that it is known and is a fixed number.

$$\lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} = \lim_{h\to 0} \frac{(t_0+h)^2 - t_0}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + 2ht_0}{h} = \lim_{h\to 0}(h+2t_0) = 2t_0.$$

Notation: Instead of h, occasionally we use Δt (or Δx, depends on the variable you use) to represent small increment of t (or x). Δt or Δx are used as a variable symbol, not Δ times x.

So the average velocity between t_0 and $t_0 + h$ is given by

$$\frac{f(t_0 + h) - f(t_0)}{h}.$$

The velocity of $f(t)$ at $t = t_0$ is given by (if the limit exists)

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

1.2 Tangent line

The rate of change of a function is the slope of the tangent line. For now, consider the following informal definition of a *tangent line*:

> Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

We illustrate this informal definition with Figure 1.

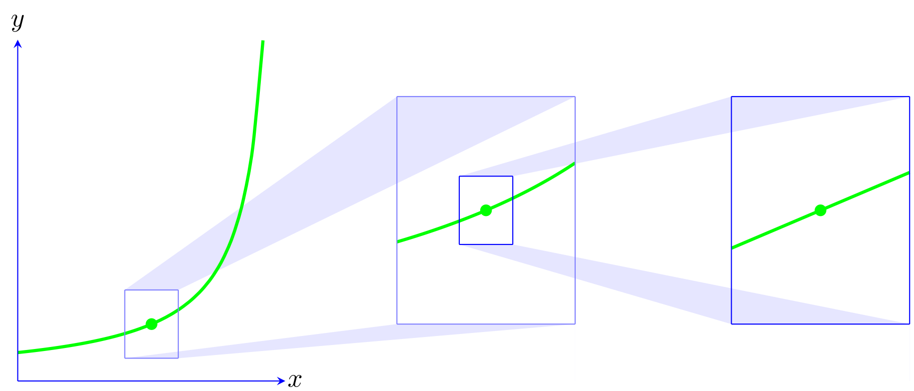

Figure 1: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x. (source: mooculus textbook)

A **secant** is a line passing through two points on a curve. The definition of **tangent** line is the limit of a secant joining two distinct points as the distance between two points tends to zero.

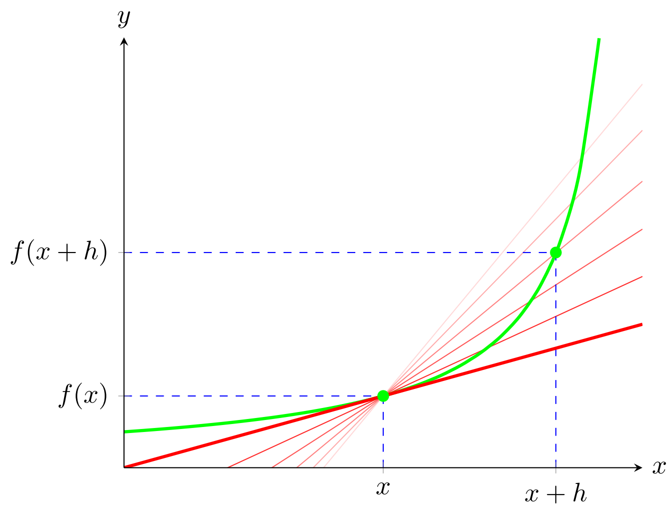

Figure 2: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$. (Source:moonculus textbook)

The slope of the secant line passes through $(x, f(x))$ and $(x+h, f(x+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}.$$

See figure 2. This leads to the following definition.

Definition 1 *The* **derivative** *of $f(x)$ is the function*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \tag{1}$$

The process of computing the derivative is called **differentiation**, *and we say that $f(x)$ is* **differentiable** *at $x = c$ if $f'(c)$ exists; that is if the limit defines $f'(x)$ exists when $x = c$.*

Remark By definition, if $f(c)$ is not well-defined, we cannot define $f'(c)$. So $f(x)$ must not be differentiable at $x = c$.

Other notations Another notation for $f'(x)$ is $\frac{df(x)}{dx}$. If we know $y = f(x)$, we can also write $f'(x)$ as $\frac{dy}{dx}$.

To substitute $x = x_0$ into the differentiation, we can write it as

$$f'(x_0)$$

or

$$\left.\frac{df(x)}{dx}\right|_{x=x_0}.$$

Remark When we said that we use **the first principle** to find derivatives, we mean that we use the definition (1) to find the derivative. However, later we will learn faster techniques to find derivatives.

Example 5 Let $f(x) = x^3$. Then (i) Find the derivative of $f(x)$. (ii) Find the equation of the tangent line to the curve at $x = -1$.

Answer (i) By the definition

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2 + 3x0 + 0^2 = 3x^2.$$

(ii) The equation of the tangent to the curve at $x = -1$ is

$$\frac{y - f(-1)}{x - (-1)} = \text{slope of the tangent line} = f'(-1) = 3(-1)^2 = 3.$$

Thus

$$\frac{y+1}{x+1} = 3$$

and hence

$$y + 1 = 3x + 3$$

or

$$y = 3x + 2.$$

Example 6 Let $f(x) = \frac{x+1}{x-1}$. Using the definition of derivative, compute $f'(x)$ for $x \neq -1$.
Answer

$$f(x+h) - f(x) = \frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}$$
$$= \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)}$$
$$= \frac{-2h}{(x-1)(x+h-1)}.$$

Therefore

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-2}{(x-1)(x+h-1)}$$
$$= \frac{\lim_{h\to 0}(-2)}{\lim_{h\to 0}(x-1)(x+h-1)} = \frac{-2}{(x-1)^2}.$$

Example 7 Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
Answer

$$\frac{d}{dx}\sqrt{x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\lim_{h\to 0}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Example 8 Find the derivative of $f(x) = \sqrt[3]{x}$.
Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
Answer

$$\begin{aligned}\frac{d}{dx}\sqrt{x} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt[3]{x+h}-\sqrt[3]{x})(\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2)}{h(\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2)} \\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2} \\ &= \frac{1}{\lim_{h\to 0}(\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2)} = \frac{1}{3\sqrt[3]{x}^2} = \frac{1}{3}x^{-\frac{2}{3}}.\end{aligned}$$

Example 9 Discuss the differentiability of $f(x) = |x|$.
Answer: If $x_0 > 0$, then

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{x_0+h-x_0}{h} = 1.$$

So $f'(x_0) = 1$. Similarly if $x_0 < 0$, $f'(x_0) = -1$. The problem is at $x_0 = 0$.

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1.$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h}{h} = -1.$$

So f is not differentiable at $x = 0$.

$$\frac{d|x|}{dx} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ \text{not differentiable} & \text{if } x = 0. \end{cases}$$

Proposition 2 *If $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.*

Proof.

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{h\to 0} f(x_0+h) \\ &= \lim_{h\to 0}(f(x_0) + h\frac{f(x_0+h)-f(x_0)}{h}) \\ &= f(x_0) + \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\right) \\ & f(x_0) + 0\times f'(x_0) = f(x_0).\end{aligned}$$

□

The converse if not true, for example, let $f(x) = |x|$. It is not differentiable at $x = 0$ but is continuous at $x = 0$.

Exercise Let

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 1 \\ 1 - x, & \text{if } x < 1 \end{cases}$$

(a) Show that $f(x)$ is continuous at $x = 1$.

(b) Show that $f(x)$ is differentiable everywhere except $x = 1$, and

$$f'(x) = \begin{cases} 2x, & \text{if } x > 1 \\ \text{undefined}, & \text{if } x = 1 \\ -1, & \text{if } x < 1 \end{cases}$$

Example 10 Discuss the differentiability of the following function at $x = 0$.

1.
$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

2.
$$g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

3.
$$h(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

Answer:

1. We are going to show that $f(x)$ is not continuous at $x = 0$ by the method in lecture 4, then hence by the previous proposition, $f(x)$ is not differentiable at $x = 0$.
   Let $a_n = \frac{1}{2n\pi}$, then $\lim_{n\to\infty} a_n = 0$. Also $f(a_n) = \sin(2n\pi) = 0$. So $\lim_{n\to\infty} f(a_n) = 0$.
   Let $b_n = \frac{1}{\frac{\pi}{2}+2n\pi}$, then $\lim_{n\to\infty} b_n = 0$. Also $f(b_n) = \sin(\frac{\pi}{2} + 2n\pi) = 1$. So $\lim_{n\to\infty} f(b_n) = 0$.
   $$\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n).$$
   By Lecture 4 Theorem 1 Method 2, $\lim_{x\to 0}\sin\frac{1}{x}$ does not exists. So it is not continuous at $x = 0$.

2. We are going to use the first principle.
   $$\lim_{x\to 0}\frac{g(x) - g(0)}{x} = \lim_{x\to 0}\sin\frac{1}{x}.$$
   we know that from the previous part that the limit diverges.

3. Again we are going to use the first principle.

4.
   $$\lim_{x\to 0}\frac{h(x) - h(0)}{x} = \lim_{x\to 0} x\sin\frac{1}{x}.$$

Because for all $x \neq 0$,

$$-|x| \leq x \sin \frac{1}{x} \leq |x|$$

and

$$\lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0.$$

So by the Sandwich theorem,

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Hence $h(x)$ is differentiable at $x = 0$. In fact $h'(0) = 0$

Definition 3 *If $f(x)$ is differentiable everywhere, then $f(x)$ is said to be a differentiable function.*